Documents de Travail du
Centre d'Economie de la Sorbonne

CES Working Papers

Possibility functions and regular economies

Jean-Marc BONNISSEAU, Elena L. Del MERCATO

2007.33

CENTRE NATIONAL
DE LA RECHERCHE
SCIENTIFIQUE

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://ces.univ-paris1.fr/cesdp/CES-docs.htm
ISSN : 1955-611X

Possibility functions and regular economies ⋆

Jean-Marc Bonnisseau [1]

Centre d'Economie de la Sorbonne, Université Paris 1

Elena L. del Mercato [2]

DISES and CSEF, Università degli Studi di Salerno

First version: March 7, 2006. Revised version: May 30, 2007.

Abstract

In the spirit of Smale's work, we consider pure exchange economies with general consumption sets. In this paper, the consumption set of each household is described in terms of a function called *possibility function*. The main innovation comes from the dependency of each possibility function with respect to the individual endowment. We prove that, generically in the space of endowments and possibility functions, economies are regular. A regular economy has only a finite number of equilibria which locally depend on endowments and possibility functions in a continuous manner.

JEL classification: C62, D11, D50.

Key words: General economic equilibrium, consumption possibility, regular economy.

⋆ Earlier versions circulated under the title "Consumption possibilities and regular economies". Presented at the Conférence Internationale sur les Mathématiques de l'Optimization et de la Décision in Guadeloupe (2006), and at the 15th European Workshop on General Equilibrium Theory in Lisbon (2006), they also benefited from the comments of these audiences.

[1] Centre d'Economie de la Sorbonne (CES), Université Paris 1 Panthéon-Sorbonne–CNRS, 106-112 Boulevard de l'Hôpital, 75647 Paris Cedex 13, France. E-mail: Jean-Marc.Bonnisseau@univ-paris1.fr.

[2] Dipartimento di Scienze Economiche e Statistiche (DISES) and Centre for Studies in Economics and Finance (CSEF), Università degli Studi di Salerno, 84084 Fisciano (SA), Italy. E-mail: edmercat@unisa.it.

1 Introduction

The objective of this paper is to study the regularity of pure exchange economies with general consumption sets depending on individual endowments.

Debreu (1970) was at the start of the global approach to equilibrium analysis from a differentiable viewpoint. This approach is based on the central concept of *regular economy.*[3] A regular economy has a finite number of equilibria and, around each equilibrium, there exists a differentiable or continuous selection of the equilibrium set with respect to the parameters describing the economy.[4]

Our work is based on the Arrow–Debreu foundation of general equilibrium theory. As in general equilibrium models à la Arrow–Debreu, each household h has to choose a consumption in his *consumption set* X_h, i.e., in the set of all consumption alternatives which are *a priori possible* for him. In Debreu (1959) for instance, one can find several motivations which lead to consider individual consumption sets. Among them, Debreu provides as example also a survival possibility to work, "...the decision for an individual to have during next year as sole input one pound of rice and as output one thousand hours of some type of labor could not be carried out."

In this paper we focus on general consumption sets depending on individual endowments. We provide below some evidences of this dependency.

- Since the individual endowment is an "indicator" of *social status*, then it affects the individual *a priori possibility* to choose. Indeed, the social status of an individual often imposes a minimal level of consumption for some commodities which is necessary to preserve his social status. This implies a lower bound on the consumption for some commodities which depends on the individual endowment. Moreover, the social status of an individual clearly affects his skills and his abilities that may impose a maximal level of consumption for some commodities. This indirectly implies an upper bound on the consumption for some commodities which depends on the individual endowment.
- In a consumption-leisure model, the consumption in leisure is bounded below by a bound which depends on the maximal possible workload of the household. This workload is precisely the individual endowment in the commodity "labor" which is the opposite of leisure.
- If some trading rules impose some limitations on the possible net trade of an household on the market, this leads to a consumption set which depends on

[3] For major and exhaustive expositions see Debreu (1983) and Mas-Colell (1985).
[4] Observe that if the parameters are elements of an arbitrary topological space, then differentiability is a meaningless idea, but one still gets continuity.

the individual endowment, since the possible consumptions are constrained as the individual endowment plus the possible net trade.

Moreover, importantly observe that the "survival condition" represents an *umbilical cord* between the endowment and the consumption set of an individual.[5] Whereas this link is quite innocuous when one just analyzes the existence of equilibria for a given economy, it plays an important role in terms of global analysis. Indeed, since altering the endowment of an individual might be also inadvertently alter his possibility to survive, above all, one may allow consumption sets which depend on individual endowments.

In this paper, we consider a pure exchange economy with a finite number of commodities and of households. Each household is characterized by an individual endowment of commodities, a *possibility function* and a utility function. We call possibility function a function describing a consumption set.[6] The main innovation comes from the dependency of the possibility function of each household with respect to his endowment. Taking prices and individual endowment as given, each household maximizes his utility function in his consumption set under his budget constraint. The definition of competitive equilibrium follows.

The main results of this paper deal with the well known regularity results given in Debreu (1970, 1976), Smale (1974a, 1974b), Allen (1981), Dierker (1982), Mas-Colell (1985) and Balasko (1988). We prove that almost all *perturbed economies* are regular and we provide the result of generic regularity in the space of endowments and possibility functions.[7]

Regularity has been studied in different contexts. Regarding analysis encompassing various sorts of constraints on individual behavior, we can quote Smale (1974a, 1974b), Villanacci (1993), Cass, Siconolfi and Villanacci (2001), Bonnisseau and Rivera Cayupi (2003, 2006), Villanacci and Zenginobuz (2005), among others.

Most related to our paper is Smale (1974b), where households have consumption sets described in terms of functions. Smale provides the result of generic regularity in the space of endowments and utility functions. Substantially, his result had to rely on perturbations of utility functions since his goal was to remove the standard hypothesis on utility functions which give rise to C^1 demand functions. Nevertheless, if utility functions satisfy assumptions which

[5] A rough version of the survival assumption states that the endowment of an individual is an interior point of his consumption set.
[6] Note that this idea is usual for smooth economies with production where each production set is described by a function called *transformation function*.
[7] Following Smale (1974a, 1974b) and Mas-Colell (1985), here almost all means in an open and full measure subset and generic means in an open and dense subset.

are standard in smooth economies, then the regularity result holds for almost all endowments. [8]

Differently from Smale (1974b), we mainly focus on consumption sets which depend on individual endowments, and we consider utility functions which satisfy standard assumptions in smooth economies (see Assumption 1).

Furthermore, in our model, the individual endowments can be outside of the consumption sets. Whereas in Smale (1974b), the individual endowments belong to the consumption sets. Indeed, in Smale (1974b), the space of endowments is the Cartesian product of the consumption sets and the consumption sets are fixed. As Mas-Colell and Smale have pointed out, "...it would make more sense to allow greater latitude for the initial allocation in the definition of economy." [9] Considering consumption sets which depend on individual endowments is also one way to answer that suggestion.

To prove non-emptiness and compactness of the equilibrium set, in Assumption 2, the assumptions on the possibility functions are adapted from del Mercato (2006a) in a natural way. [10]

The analysis in Debreu (1970, 1976), Smale (1974a, 1974b), Dierker (1982), Mas-Colell (1985) and Balasko (1988), makes appear that classical differentiability and regularity results hold whenever, at equilibrium, all agents are in the interior of their consumption sets. Since nothing prevents the equilibrium allocations to be on the boundary of the consumption sets, then to prove generic differentiability and regularity results, we follow the strategy laid out in Cass, Siconolfi and Villanacci (2001), where it is given a general method for encompassing individual portfolio constraints while still permitting differential techniques. [11] But the dependency of each possibility function with respect to the individual endowment leads to technical difficulties. For that reason, we consider *simple perturbations* of the possibility functions. Actually, the space of perturbations is a finite dimensional space. This is the key idea for our treatment.

Besides, we need an additional assumption on the possibility functions, namely Assumption 3, which covers three economically meaningful cases. That is, Assumption 3 holds true when the possibility function of at least one household

[8] See also Smale (1974a), Theorem 2, p. 3, and del Mercato (2006b).
[9] Smale (1974b), p. 123.
[10] In del Mercato (2006a), utility and possibility functions also depend on the consumptions of all households.
[11] Many different authors have considered restrictions on the markets, Balasko, Cass and Siconolfi (1990), Cass (1990), Polemarchakis and Siconolfi (1997), Cass, Siconolfi and Villanacci (2001). The difference with our approach is that in these papers the restriction is on financial markets.

satisfies one among the following conditions: 1. it does not depend on the endowment, 2. it depends on the net trade, 3. it is additively separable in consumption and endowment.

The paper is organized as follows. Section 2 is devoted to our basic model and assumptions. In Section 3, we present the definitions of competitive equilibrium and of equilibrium function. Then, Theorem 7 provides non-emptiness and compactness results for the equilibrium set. In Section 4, we state the definitions of regular economy and of perturbed economy, and we present the main results of this paper, namely Theorem 13, Corollary 14 and Proposition 15. Theorem 13 states the result of regularity for almost all perturbed economies. Corollary 14 provides the result of generic regularity in the space of endowments and possibility functions. From Proposition 15, one deduces that a regular economy has a finite number of equilibria which locally depend on endowments and possibility functions in a continuous manner. In Section 5, we prove Theorem 13 and Corollary 14. In particular, in Subsections 5.1 and 5.2, the strategy of the proof for Theorem 13 is detailed, and in Subsection 5.3, we show Corollary 14. All the proofs are gathered in Appendix A, except for Theorem 13 and Corollary 14. In Appendix B, the reader can find results from differential topology used in our analysis.

2 The model and the assumptions

There are $C < \infty$ physical commodities labeled by superscript $c \in \{1, ..., C\}$. The commodity space is $\mathbb{R}^C_{++}$. There are $H < \infty$ households labeled by subscript $h \in \mathcal{H} := \{1, ..., H\}$. Each household $h \in \mathcal{H}$ is characterized by an individual endowment of commodities, a possibility function and a utility function. The possibility function of household h depends on his endowment.

The notations are summarized below.

- x^c_h is the consumption of commodity c by household h;
  $x_h := (x^1_h, .., x^c_h, .., x^C_h) \in \mathbb{R}^C_{++}$; $x := (x_h)_{h\in\mathcal{H}} \in \mathbb{R}^{CH}_{++}$.
- e^c_h is the endowment of commodity c owned by household h;
  $e_h := (e^1_h, .., e^c_h, .., e^C_h) \in \mathbb{R}^C_{++}$; $e := (e_h)_{h\in\mathcal{H}} \in \mathbb{R}^{CH}_{++}$.
- As in general equilibrium model à la Arrow–Debreu, each household h has to choose a consumption in his consumption set X_h, i.e., in the set of all consumption alternatives which are *a priori possible* for him. In our paper, in the spirit of Smale's work (1974b), the consumption set of household h is described in terms of a function χ_h. Observe that this idea is usual for smooth economies with production where each production set is described by a function called *transformation function*. We call χ_h *possibility func-*

tion.[12]

The main innovation of this paper comes from the dependency of the possibility function of each household with respect to his individual endowment. That is, given $e_h \in \mathbb{R}^C_{++}$, the consumption set of household h is the following set,

$$X_h(e_h) = \left\{ x_h \in \mathbb{R}^C_{++} : \chi_h(x_h, e_h) \geq 0 \right\}$$

where $\chi_h : \mathbb{R}^C_{++} \times \mathbb{R}^C_{++} \to \mathbb{R}$; $\chi := (\chi_h)_{h \in \mathcal{H}}$.

- Each household $h \in \mathcal{H}$ has preferences described by a utility function u_h from $\mathbb{R}^C_{++}$ to $\mathbb{R}$, and $u_h(x_h) \in \mathbb{R}$ is the utility of household h associated with the consumption x_h; $u := (u_h)_{h \in \mathcal{H}}$.
- $\mathcal{E} := (e, \chi, u)$ is an economy.
- p^c is the price of one unit of commodity c; $p := (p^1, .., p^c, .., p^C) \in \mathbb{R}^C_{++}$.
- Given a vector $w = (w^1, .., w^c, .., w^C) \in \mathbb{R}^C$, we denote

$$w^\setminus := (w^1, .., w^c, .., w^{C-1}) \in \mathbb{R}^{C-1}$$

- Let w and v be two vectors in $\mathbb{R}^C$, A be a real matrix with R rows and C columns, and B be a real matrix with C rows and L columns. wv denotes the *scalar product* of w and v; $A \cdot B$ denotes the *matrix product* of A and B; treating w as a matrix with 1 row and C columns, $w \cdot B$ denotes the matrix product of w and B. Moreover, without loss of generality A denotes both the matrix and the following linear application,

$$A : v \in \mathbb{R}^C \longrightarrow A(v) := A \cdot v^T \in \mathbb{R}^R$$

  where, treating v as a matrix with 1 row and C columns, v^T is the transpose of v. When $R = 1$, $A(v)$ coincides with the scalar product Av of A and v treating A and v as vectors in $\mathbb{R}^C$.

From now on, we make the following assumptions on (χ, u). The assumptions on u_h are standard in smooth general equilibrium models. The assumptions on χ_h are adapted from del Mercato (2006a) in a natural way.

Assumption 1 *For all $h \in \mathcal{H}$,*

(1) u_h *is a* C^2 *function.*
(2) u_h *is differentiably strictly increasing, i.e., for every* $x_h \in \mathbb{R}^C_{++}$, $D_{x_h} u_h(x_h) \in \mathbb{R}^C_{++}$.
(3) u_h *is differentiably strictly quasi-concave, i.e., for every* $x_h \in \mathbb{R}^C_{++}$, $D^2_{x_h} u_h(x_h)$ *is negative definite on* $\operatorname{Ker} D_{x_h} u_h(x_h)$.
(4) For every $u \in \operatorname{Im} u_h$, $cl_{\mathbb{R}^C} \{x_h \in \mathbb{R}^C_{++} : u_h(x_h) \geq u\} \subseteq \mathbb{R}^C_{++}$.

[12] Note that in Smale (1974b), each consumption set is described by more than one function. Our results can be extended to this case, but this is not our main objective.

Assumption 2 *Let $e \in \mathbb{R}^{CH}_{++}$. For all $h \in \mathcal{H}$,*

(1) χ_h is a C^2 function.
(2) (Convexity of the consumption set) The function $\chi_h(\cdot, e_h)$ is quasi-concave.[13]
(3) (Survival assumption) There exists $\widetilde{x}_h \in \mathbb{R}^C_{++}$ such that $\chi_h(\widetilde{x}_h, e_h) > 0$ and $\widetilde{x}_h \ll e_h$.
(4) (Non-satiation) For every $x_h \in \mathbb{R}^C_{++}$, (a) $D_{x_h}\chi_h(x_h, e_h) \neq 0$, and (b) $D_{x_h}\chi_h(x_h, e_h) \notin -\mathbb{R}^C_{++}$.
(5) (Global desirability) For each $x \in \mathbb{R}^{CH}_{++}$ and for each $c \in \{1, ..., C\}$ there exists $h(c) \in \mathcal{H}$ such that $D_{x^c_{h(c)}}\chi_{h(c)}(x_{h(c)}, e_{h(c)}) \in \mathbb{R}_+$.

From points 1 and 2 of Assumption 2, the usual assumptions on closedness and convexity of the consumption set hold true.

Point 3 of Assumption 2 corresponds to the *survival assumption*. A rough version of the survival assumption states that the endowment of a household is an interior point of his consumption set. Point 3 of Assumption 2 is a "survival condition" even if e_h is not assumed to belong to the consumption set $X_h(e_h)$. Since commodity prices are strictly positive, as a consequence of point 3 of Assumption 2 we have that, given the endowment $e_h \in \mathbb{R}^C_{++}$, the consumption set $X_h(e_h)$ has non-empty intersection with the budget set.

Condition $4a$ in Assumption 2 means that, given $e_h \in \mathbb{R}^C_{++}$, the *possibility surfaces* $[\chi_h(\cdot, e_h)]^{-1}(r)$ are smooth as r varies on $\operatorname{Im}\chi_h(\cdot, e_h)$. Moreover, the possibility for each household to be locally "non-satiated" remaining in his consumption set is crucial in order to expect existence of equilibria. According to point 2 of Assumption 1, it holds true if each household is "free" for at least one good, i.e., given $e_h \in \mathbb{R}^C_{++}$, there is at least one good c such that $D_{x^c_h}\chi_h(x_h, e_h) = 0$ for every $x_h \in \mathbb{R}^C_{++}$. Note that point $4b$ of Assumption 2 is weaker than the previous condition.

Finally, commodities must be *desirable*. The idea is the following: a good is "globally desirable" if for each consumption configuration of the economy there is one household that can locally strictly increase his utility by consuming a larger quantity of this good remaining in his consumption set. According to point 2 of Assumption 1, it is trivially true that each good is desirable if there is at least one *super-household*, i.e., a household who is not constrained in consumption possibilities, that is his consumption set is $\mathbb{R}^C_{++}$. The concept of global desirability, mentioned above, is stronger than what we actually require in point 5 of Assumption 2.

The above assumptions are enough to get the existence of competitive equilibria for a given economy. But, for our purpose, that is the generic regularity

[13] Since χ_h is C^2, we have that for every $x_h \in \mathbb{R}^C_{++}$, $D^2_{x_h}\chi_h(x_h, e_h)$ is negative semidefinite on $\operatorname{Ker} D_{x_h}\chi_h(x_h, e_h)$.

of economies, we need an additional assumption on the possibility functions.

Assumption 3 *Let* $e \in \mathbb{R}^{CH}_{++}$.

For each $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CH}$ *such that* $y_h \in \operatorname{Ker} D_{x_h}\chi_h(x_h, e_h)$ *for all* $h \in \mathcal{H}$, *there is* $k \in \mathcal{H}$ *such that* $\left[y_k \cdot \left(D^2_{x_k}\chi_k(x_k, e_k) + D^2_{e_k x_k}\chi_k(x_k, e_k)\right)\right] y_k \leq 0.$

Observe that if we skip the term $D^2_{e_k x_k}\chi_k(x_k, e_k)$, then we go back to point 2 of Assumption 2. Next, we provide three relevant cases in which Assumption 3 is satisfied.

(1) If one consumption set does not depend on the individual endowment, i.e., one possibility function χ_k does not depend on e_k, then point 2 of Assumption 2 implies that Assumption 3 is satisfied.
   Then, our analysis encompasses the case analyzed by Debreu in which all the consumption sets coincide with $\mathbb{R}^{C}_{++}$.[14]
(2) If one possibility function χ_k depends on the net trade, i.e., it takes the following form $\chi_k(x_k, e_k) := \widetilde{\chi}_k(x_k - e_k)$, then Assumption 3 holds true.
(3) If one possibility function χ_k is additively separable in consumption and endowment, i.e., it takes the following form $\chi_k(x_k, e_k) := \widetilde{\chi}_k(x_k) + \phi_k(e_k)$, then point 2 of Assumption 2 implies that Assumption 3 is satisfied.

3 Competitive equilibrium

First, we adapt to the previous model the standard competitive equilibrium concept usually defined for a pure exchange economy à la Arrow–Debreu. Second, we characterize in (3) the solution of household h's maximization problem in terms of Kuhn–Tucker conditions. Then, in Remark 6 we restate equilibria in terms of solutions of equations, from which we deduce in (4) the *equilibrium function* $F_{\mathcal{E}}$ for a given economy $\mathcal{E}$. Finally, Theorem 7 states the non-emptiness and the compactness of the equilibrium set. All the proofs of the results stated in this section are gathered in Appendix A.

Without loss of generality, commodity C is the *numeraire good*. Then, given $p^\backslash \in \mathbb{R}^{C-1}_{++}$ with innocuous abuse of notation we denote $p := (p^\backslash, 1) \in \mathbb{R}^{C}_{++}$.

Definition 4 $(x^*, p^{*\backslash}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{C-1}_{++}$ *is a competitive equilibrium for* $\mathcal{E}$ *if*

[14] Observe that in Smale (1974b), all the consumption sets do not depend on individual endowments. But, also note that in Smale (1974b), the vector of individual endowments has to belong to the Cartesian product of the consumption sets. Whereas in our assumptions, $e \in \mathbb{R}^{CH}_{++}$.

- *for all $h \in \mathcal{H}$, x_h^* solves the following problem*

$$\begin{aligned} \max_{x_h \in \mathbb{R}^C_{++}} \quad & u_h(x_h) \\ \text{subject to } & \chi_h(x_h, e_h) \geq 0 \\ & p^* x_h \leq p^* e_h \end{aligned} \tag{1}$$

- *x^* satisfies market clearing conditions*

$$\sum_{h \in \mathcal{H}} x_h^* = \sum_{h \in \mathcal{H}} e_h \tag{2}$$

Proposition 5 *Let $\mathcal{E}$ be an economy satisfying Assumptions 1 and 2, and $p^{*\backslash} \in \mathbb{R}^{C-1}_{++}$. Problem (1) has a unique solution. $x_h^* \in \mathbb{R}^C_{++}$ is the solution to problem (1) if and only if there exists $(\lambda_h^*, \mu_h^*) \in \mathbb{R}_{++} \times \mathbb{R}$ such that $(x_h^*, \lambda_h^*, \mu_h^*)$ is the unique solution of the following system.*

$$\begin{cases} (h.1) \; D_{x_h} u_h(x_h) - \lambda_h p^* + \mu_h D_{x_h} \chi_h(x_h, e_h) = 0 \\ (h.2) \; -p^*(x_h - e_h) = 0 \\ (h.3) \; \min\{\mu_h, \chi_h(x_h, e_h)\} = 0 \end{cases} \tag{3}$$

Define the set of endogenous variables as $\Xi := (\mathbb{R}^C_{++} \times \mathbb{R}_{++} \times \mathbb{R})^H \times \mathbb{R}^{C-1}_{++}$, with generic element $\xi := (x, \lambda, \mu, p^\backslash) := ((x_h, \lambda_h, \mu_h)_{h \in \mathcal{H}}, p^\backslash)$. We can now describe *extended equilibria* using system (3) and market clearing conditions (2). Observe that, from Definition 4 and Proposition 5, the market clearing condition for good C is "redundant" (see equations $(h.2)_{h \in \mathcal{H}}$ in (3)). Therefore, in the following remark we omit in (2) the condition for good C.

Remark 6 *Let $\mathcal{E}$ be an economy satisfying Assumptions 1 and 2, $\xi^* \in \Xi$ is an extended competitive equilibrium for $\mathcal{E}$ if and only if $(x_h^*, \lambda_h^*, \mu_h^*)$ solves system (3) at $p^{*\backslash}$ for all $h \in \mathcal{H}$, and $\sum_{h \in \mathcal{H}} (x_h^{*\backslash} - e_h^{*\backslash}) = 0$. With innocuous abuse of terminology, we call ξ^* simply an equilibrium.*

For a given economy $\mathcal{E}$, define the equilibrium function $F_{\mathcal{E}} : \Xi \longrightarrow \mathbb{R}^{\dim \Xi}$

$$F_{\mathcal{E}}(\xi) := ((F_{\mathcal{E}}^{h.1}(\xi), F_{\mathcal{E}}^{h.2}(\xi), F_{\mathcal{E}}^{h.3}(\xi))_{h \in \mathcal{H}}, F_{\mathcal{E}}^{M}(\xi)) \tag{4}$$

where

$F_{\mathcal{E}}^{h.1}(\xi) := D_{x_h} u_h(x_h) - \lambda_h p + \mu_h D_{x_h} \chi_h(x_h, e_h)$, $F_{\mathcal{E}}^{h.2}(\xi) := -p(x_h - e_h)$, $F_{\mathcal{E}}^{h.3}(\xi) := \min\{\mu_h, \chi_h(x_h, e_h)\}$, and $F_{\mathcal{E}}^{M}(\xi) := \sum_{h \in \mathcal{H}} (x_h^\backslash - e_h^\backslash)$.

From Remark 6, $\xi^* \in \Xi$ is an equilibrium for $\mathcal{E}$ if and only if $F_{\mathcal{E}}(\xi^*) = 0$. The following theorem provides non-emptiness and compactness results for the equilibrium set $F_{\mathcal{E}}^{-1}(0)$.

Theorem 7 *For each economy $\mathcal{E}$ satisfying Assumptions 1 and 2, the set of equilibria for $\mathcal{E}$ is non-empty and compact.*

4 Regular economies and possibility perturbations

Let us begin with the definition of *regular economy.*

Definition 8 *$\mathcal{E}$ is a regular economy if*

(1) for each $\xi^* \in F_{\mathcal{E}}^{-1}(0)$, $F_{\mathcal{E}}$ is differentiable at ξ^*, and
(2) 0 is a regular value for $F_{\mathcal{E}}$, which means that for each $\xi^* \in F_{\mathcal{E}}^{-1}(0)$, the differential mapping $D_\xi F_{\mathcal{E}}(\xi^*)$ is onto.

Our analysis is based on results from differential topology, in the spirit of Debreu (1970, 1976), Smale (1974a, 1974b), Dierker (1982), Mas-Colell (1985), and Balasko (1988). Their analysis shows that the differentiability of $F_{\mathcal{E}}$ holds whenever, at equilibrium, all agents are in the interior of their consumption sets. Since nothing prevents the equilibrium allocations to be on the boundary of the consumption sets, for each $h \in \mathcal{H}$ the function

$$F_{\mathcal{E}}^{h.3}(\xi) = \min\{\mu_h, \chi_h(x_h, e_h)\}$$

is not everywhere differentiable. Then, first of all, it shall be show that, generically, the equilibrium function is differentiable at each equilibrium.

For that purpose, observe that in Cass, Siconolfi and Villanacci (2001), it is given a general method for encompassing individual portfolio constraints while still permitting differential techniques. Then, to prove generic differentiability and regularity results we follow the strategy laid out by Cass, Siconolfi and Villanacci (2001).[15] But the dependency of each χ_h with respect to the individual endowment leads to technical difficulties. For that reason, we consider *simple perturbations* of the possibility functions. Actually, the space of perturbations is a finite dimensional space. This is the key idea for our treatment.

The main results of this section are Theorem 13, Corollary 14 and Proposition 15. Theorem 13 and Corollary 14 embody the regularity results. Theorem 13 states the result of regularity for almost all *perturbed economies.* Corollary 14 provides the result of generic regularity in the space of endowments and

[15] The reader can find a survey of this approach in Villanacci et al. (2002).

possibility functions. Proposition 15 is a consequence of Corollary 14 and of the Implicit Function Theorem. From Proposition 15, one deduces that a regular economy has a finite number of equilibria which locally depend on endowments and possibility functions in a continuous manner.

The sets of utility and possibility functions that we consider are defined below.

Definition 9 *The set of utility functions $u = (u_h)_{h\in\mathcal{H}}$ which satisfy Assumption 1 is denoted by $\mathcal{U}$. The set of possibility functions $\chi = (\chi_h)_{h\in\mathcal{H}}$ which satisfy Assumptions 2 and 3 for every $e \in \mathbb{R}^C_{++}$ is denoted by $\mathcal{X}$.*

To state Theorem 13 we need introduce the definition of *perturbed economy.* In the following definitions and in Theorem 13, we take for fixed an arbitrary $(\chi, u) \in \mathcal{X} \times \mathcal{U}$.

Definition 10 *A perturbed economy associated with endowments $e \in \mathbb{R}^{CH}_{++}$ and possibility levels $a = (a_h)_{h\in\mathcal{H}} \in \mathbb{R}^H$ is defined by $\mathcal{E}^a := (e, \chi^a, u)$, where $\chi^a := (\chi^a_h)_{h\in\mathcal{H}}$ and for every $h \in \mathcal{H}$,*

$$\chi^a_h := \chi_h + a_h$$

is the perturbed possibility function.

In the following definition, we consider perturbed economies for which the survival assumption holds true.[16]

Definition 11 *A perturbed economy $\mathcal{E}^a = (e, \chi^a, u)$ is parameterized by endowments and possibility levels taken in the following set.*

$$\Lambda := \left\{ (e,a) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^H \;\middle|\; \begin{array}{l} \forall h \in \mathcal{H},\ \exists\, \widetilde{x}_h \in \mathbb{R}^C_{++} \text{ such that} \\ \chi_h(\widetilde{x}_h, e_h) + a_h > 0 \text{ and } \widetilde{x}_h \ll e_h \end{array} \right\}$$

Remark 12 *Importantly, observe that*

- *by point 3 of Assumption 2, $\mathbb{R}^{CH}_{++} \times \mathbb{R}^H_{+} \subseteq \Lambda$. Then, $(e, 0) \in \Lambda$ for every $e \in \mathbb{R}^{CH}_{++}$. This means that the set Λ embodies every economy $\mathcal{E} = (e, \chi, u) \in \mathbb{R}^{CH}_{++} \times \mathcal{X} \times \mathcal{U}$.*
- *By points 1 and 3 of Assumption 2, Λ is an open subset of $\mathbb{R}^{CH}_{++} \times \mathbb{R}^H$.*
- *The perturbed possibility functions χ^a satisfy Assumptions 2 and 3 for each $(e, a) \in \Lambda$.*

We can now state the result of regularity for almost all perturbed economies.

Theorem 13 *The set Λ^r of $(e, a) \in \Lambda$ such that $\mathcal{E}^a = (e, \chi^a, u)$ is regular is an open and full measure subset of Λ.*

[16] See point 3 of Assumption 2.

Now, endow the set $\mathcal{X}$ with the topology induced by the product topology of the topology of the C^2 uniform convergence on compacta (see Appendix B). As a consequence of Theorem 13 we obtain the following corollary which provides the result of generic regularity.

Corollary 14 *For each $u \in \mathcal{U}$, the set $\mathcal{R}_u$ of $(e, \chi) \in \mathbb{R}^{CH}_{++} \times \mathcal{X}$ such that $\mathcal{E} = (e, \chi, u)$ is a regular economy is an open and dense subset of $\mathbb{R}^{CH}_{++} \times \mathcal{X}$.*

Finally, take for fixed an arbitrary $u \in \mathcal{U}$ and define the following *global equilibrium function* using the equilibrium function defined in (4).

$$F : (\xi, e, \chi) \in \Xi \times \mathbb{R}^{CH}_{++} \times \mathcal{X} \longrightarrow F(\xi, e, \chi) := F_{\mathcal{E}}(\xi) \in \mathbb{R}^{\dim \Xi} \tag{5}$$

As a direct consequence of Corollary 14 and of the Implicit Function Theorem (see Theorem 35 in Appendix B), we obtain the following proposition which provides the main properties of regular economies.

Proposition 15 *Let $u \in \mathcal{U}$. For each regular economy $\mathcal{E} = (e, \chi, u)$,*

(1) there exits $r \in \mathbb{N}$ such that $F_{\mathcal{E}}^{-1}(0) = \{\xi^1, ..., \xi^r\}$.

(2) There exist an open neighborhood I of (e, χ) in $\mathbb{R}^{CH}_{++} \times \mathcal{X}$, and for each $i = 1, ..., r$ an open neighborhood N_i of ξ^i in Ξ and a continuous function $g_i : I \to N_i$ such that

- *(a) $N_j \cap N_k = \emptyset$ if $j \neq k$,*
- *(b) $g_i(e, \chi) = \xi^i$,*
- *(c) $F(\xi', e', \chi') = 0$ holds for $\xi' \in N_i$ and for $(e', \chi') \in I$ if and only if $\xi' = g_i(e', \chi')$,*
- *(d) also the economies $\mathcal{E}' = (e', \chi', u)$ with $(e', \chi') \in I$ are regular economies.*

5 Proofs

In this section, we prove Theorem 13 and Corollary 14. In Subsection 5.1, substantially, we prove that the equilibrium function is differentiable at each equilibrium allocation for almost all perturbed economies. In Subsection 5.2, using the results obtained in Subsection 5.1, the strategy of the proof for Theorem 13 is detailed. Finally, in Subsection 5.3 we prove Corollary 14 using several lemmas. All the proofs of the results stated in this section are gathered in Appendix A.

In this subsection, first, we define in (6) the equilibrium function for perturbed economies and we present Proposition 16 which provides non-emptiness and compactness results for each perturbed economy. Second, we state the definition of *border line case* for perturbed economies, i.e., a situation in which, at equilibrium, a consumption is on the boundary of the consumption set and the associated Lagrange multiplier vanishes. Finally, we present Proposition 19 which is the main result of this subsection. Especially, we deduce from Proposition 19 that the equilibrium function is differentiable at each equilibrium allocation for almost all perturbed economies. The proof of Proposition 19 is built upon Lemmas 17 and 18.

Take for fixed $(\chi, u) \in \mathcal{X} \times \mathcal{U}$ and consider the set Λ given in Definition 11. Define the equilibrium function for perturbed economies $\widetilde{F} : \Xi \times \Lambda \longrightarrow \mathbb{R}^{\dim \Xi}$

$$\widetilde{F}(\xi, e, a) := ((\widetilde{F}^{h.1}(\xi, e, a), \widetilde{F}^{h.2}(\xi, e, a), \widetilde{F}^{h.3}(\xi, e, a))_{h \in \mathcal{H}}, \widetilde{F}^{M}(\xi, e, a)) \quad (6)$$

where

$\widetilde{F}^{h.1}(\xi, e, a) := D_{x_h} u_h(x_h) - \lambda_h p + \mu_h D_{x_h} \chi_h(x_h, e_h)$, $\widetilde{F}^{h.2}(\xi, e, a) := -p(x_h - e_h)$, $\widetilde{F}^{h.3}(\xi, e, a) := \min\{\mu_h, \chi_h(x_h, e_h) + a_h\}$, and $\widetilde{F}^{M}(\xi, e, a) := \sum_{h \in \mathcal{H}} (x_h^{\backslash} - e_h^{\backslash})$,

and observe that for each $(e, a) \in \Lambda$, the following function

$$\widetilde{F}_{e,a} : \xi \in \Xi \longrightarrow \widetilde{F}_{e,a}(\xi) := \widetilde{F}(\xi, e, a) \in \mathbb{R}^{\dim \Xi}$$

is nothing else than the function defined in (4) for the economy $\mathcal{E}^a = (e, \chi^a, u)$.

Proposition 16 *For each $(e, a) \in \Lambda$, $\widetilde{F}_{e,a}^{-1}(0)$ is non-empty and compact.*

Given $(\xi, e, a) \in \widetilde{F}^{-1}(0)$, we say that household h is at a *border line case* if $\mu_h = \chi_h(x_h, e_h) + a_h = 0$. The main result of this subsection is Proposition 19 stating that border line cases occur outside an open and full measure subset Θ of the space Λ. To construct the set Θ and to prove that Θ is open and full measure subset of Λ we need introduce some preliminary definitions and lemmas. Define

$$B_h := \left\{(\xi, e, a) \in \widetilde{F}^{-1}(0) : \ \mu_h = \chi_h(x_h, e_h) + a_h = 0\right\} \text{ and } B := \bigcup_{h \in \mathcal{H}} B_h$$

B_h is closed in $\widetilde{F}^{-1}(0)$ for each $h \in \mathcal{H}$, then B is closed in $\widetilde{F}^{-1}(0)$. Define also the restriction to $\widetilde{F}^{-1}(0)$ of the projection of $\Xi \times \Lambda$ onto Λ,

$$\Phi : (\xi, e, a) \in \widetilde{F}^{-1}(0) \to \Phi(\xi, e, a) := (e, a) \in \Lambda$$

and

$$\Theta := \Lambda \setminus \Phi(B) \tag{7}$$

By definition, for each $(\xi, e, a) \in \widetilde{F}^{-1}(0) \cap (\Xi \times \Theta)$ and for each $h \in \mathcal{H}$, either

$$\mu_h > 0 \text{ or } \chi_h(x_h, e_h) + a_h > 0$$

We have to prove that $\Phi(B)$ is closed and of measure zero in Λ.

The closedness of $\Phi(B)$ follows from the closedness of B in $\widetilde{F}^{-1}(0)$ and from the properness of Φ obtained by the following lemma.[17]

Lemma 17 *The function Φ is proper.*

To show that $\Phi(B)$ is of measure zero, define

$$\mathcal{P} := \left\{ \mathcal{J} = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\} \;\middle|\; \begin{array}{l} \mathcal{H}_i \subseteq \mathcal{H},\ \forall i = 1,2,3; \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3 = \mathcal{H}; \\ \mathcal{H}_i \cap \mathcal{H}_j = \emptyset,\ \forall i,j = 1,2,3, i \neq j;\ \text{and } \mathcal{H}_3 \neq \emptyset \end{array} \right\}$$

Let $\mathcal{J} = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\} \in \mathcal{P}$, for each $i = 1, 2, 3$ denote by $\mathcal{H}_i(\mathcal{J})$ the set $\mathcal{H}_i$ in $\mathcal{J}$, and by $|\mathcal{H}_i(\mathcal{J})|$ the number of element of $\mathcal{H}_i(\mathcal{J})$. Define

$$\Xi_{\mathcal{J}} := \mathbb{R}_{++}^{(C+1)H} \times (\mathbb{R}^{|\mathcal{H}_1(\mathcal{J})| + |\mathcal{H}_3(\mathcal{J})|} \times \mathbb{R}_{++}^{|\mathcal{H}_2(\mathcal{J})|}) \times \mathbb{R}_{++}^{(C-1)}$$

Observe that $\dim \Xi_{\mathcal{J}} = \dim \Xi$. Define the function $\widetilde{F}_{\mathcal{J}} : \Xi_{\mathcal{J}} \times \Lambda \to \mathbb{R}^{\dim \Xi_{\mathcal{J}}}$

$$\widetilde{F}_{\mathcal{J}}(\xi, e, a) := ((\widetilde{F}^{h.1}(\xi, e, a), \widetilde{F}^{h.2}(\xi, e, a), \widetilde{F}^{h.3}_{\mathcal{J}}(\xi, e, a))_{h \in \mathcal{H}}, \widetilde{F}^{M}(\xi, e, a))$$

where $\widetilde{F}_{\mathcal{J}}$ differs from $\widetilde{F}$ defined in (6), for the domain and for the component $\widetilde{F}^{h.3}_{\mathcal{J}}$ defined below

$$\widetilde{F}^{h.3}_{\mathcal{J}}(\xi, e, a) := \begin{cases} \mu_h \text{ if } h \in \mathcal{H}_1(\mathcal{J}) \cup \mathcal{H}_3(\mathcal{J}), \\ \chi_h(x_h, e_h) + a_h \text{ if } h \in \mathcal{H}_2(\mathcal{J}) \end{cases}$$

Moreover, given $\mathcal{J} \in \mathcal{P}$ define the set

$$\Theta_{\mathcal{J}} := \{(\xi, e, a) \in \widetilde{F}^{-1}_{\mathcal{J}}(0) : \chi_h(x_h, e_h) + a_h = 0,\ \forall h \in \mathcal{H}_3(\mathcal{J})\}$$

Given an arbitrary $(\xi, e, a) \in B$, we can define endogenously

$$\mathcal{J}(\xi, e, a) := \{\mathcal{H}_1(\xi, e, a), \mathcal{H}_2(\xi, e, a), \mathcal{H}_3(\xi, e, a)\} \in \mathcal{P} \text{ with}$$

$\mathcal{H}_1(\xi, e, a) := \{h \in \mathcal{H} :\ \mu_h = 0 \text{ and } \chi_h(x_h, e_h) + a_h > 0\}$,
$\mathcal{H}_2(\xi, e, a) := \{h \in \mathcal{H} :\ \mu_h > 0 \text{ and } \chi_h(x_h, e_h) + a_h = 0\}$,
$\mathcal{H}_3(\xi, e, a) := \{h \in \mathcal{H} :\ \mu_h = \chi_h(x_h, e_h) + a_h = 0\}$.

[17] Also see the definition of proper function, i.e., Definition 33 in Appendix B.

Then $(\xi, e, a) \in \Theta_{\mathcal{J}(\xi,e,a)}$, and we get

$$\Phi(B) \subseteq \bigcup_{\mathcal{J}\in\mathcal{P}} \Phi(\Theta_{\mathcal{J}}) \tag{8}$$

Since the number of sets involved in the above union in finite, to show that $\Phi(B)$ is of measure zero it is enough to show that $\Phi(\Theta_{\mathcal{J}})$ is of measure zero, for each $\mathcal{J} \in \mathcal{P}$. To show that $\Phi(\Theta_{\mathcal{J}})$ is of measure zero for each $\mathcal{J} \in \mathcal{P}$, we need the following definitions and the following key lemma. Given $\mathcal{J} \in \mathcal{P}$, for each $\bar{h} \in \mathcal{H}_3(\mathcal{J})$ define the function $\widetilde{F}_{\mathcal{J},\bar{h}} : \Xi_{\mathcal{J}} \times \Lambda \to \mathbb{R}^{\dim \Xi_{\mathcal{J}}+1}$

$$\widetilde{F}_{\mathcal{J},\bar{h}}(\xi, e, a) := (\widetilde{F}_{\mathcal{J}}(\xi, e, a), \widetilde{F}^{\bar{h}.4}_{\mathcal{J}}(\xi, e, a)) \tag{9}$$

where $\widetilde{F}^{\bar{h}.4}_{\mathcal{J}}(\xi, e, a) := \chi_{\bar{h}}(x_{\bar{h}}, e_{\bar{h}}) + a_{\bar{h}}$. Moreover, for each $(e, a) \in \Lambda$, define the function $\widetilde{F}_{\mathcal{J},\bar{h},e,a} : \xi \in \Xi_{\mathcal{J}} \to \widetilde{F}_{\mathcal{J},\bar{h},e,a}(\xi) := \widetilde{F}_{\mathcal{J},\bar{h}}(\xi, e, a) \in \mathbb{R}^{\dim \Xi_{\mathcal{J}}+1}$. Observe that for each $\mathcal{J} \in \mathcal{P}$, for each $\bar{h} \in \mathcal{H}_3(\mathcal{J})$ and for each $(e, a) \in \Lambda$, $\widetilde{F}_{\mathcal{J},\bar{h}}$ and $\widetilde{F}_{\mathcal{J},\bar{h},e,a}$ are differentiable on all their domain.

Lemma 18 *For each $\mathcal{J} \in \mathcal{P}$ and for each $\bar{h} \in \mathcal{H}_3(\mathcal{J})$, 0 is a regular value for $\widetilde{F}_{\mathcal{J},\bar{h}}$.*

Then, from results of differential topology and Sard's Theorem (see Theorems 30 and 32 in Appendix B), given $\mathcal{J} \in \mathcal{P}$, for each $\bar{h} \in \mathcal{H}_3(\mathcal{J})$ there exists a full measure subset $\Omega_{\mathcal{J},\bar{h}}$ of Λ such that for each $(e, a) \in \Omega_{\mathcal{J},\bar{h}}$, $\widetilde{F}^{-1}_{\mathcal{J},\bar{h},e,a}(0) = \emptyset$. Given $\mathcal{J} \in \mathcal{P}$, let

$$\Omega_{\mathcal{J}} := \bigcup_{\bar{h}\in\mathcal{H}_3(\mathcal{J})} \Omega_{\mathcal{J},\bar{h}}$$

$\Omega_{\mathcal{J}}$ is a full measure subset of Λ. Let $(e, a) \in \Omega_{\mathcal{J}}$, by definition we have that there exists $\bar{h} \in \mathcal{H}_3(\mathcal{J})$ such that $\widetilde{F}^{-1}_{\mathcal{J},\bar{h},e,a}(0) = \emptyset$. If $(e, a) \in \Phi(\Theta_{\mathcal{J}})$, by Proposition 16, there exists $\xi \in \Xi_{\mathcal{J}}$ such that $\xi \in \widetilde{F}^{-1}_{\mathcal{J},h,e,a}(0)$ for each $h \in \mathcal{H}_3(\mathcal{J})$, and we get a contradiction. Then,

$$\Phi(\Theta_{\mathcal{J}}) \subseteq \Lambda \setminus \Omega_{\mathcal{J}}$$

Since $\Lambda \setminus \Omega_{\mathcal{J}}$ is of measure zero, $\Phi(\Theta_{\mathcal{J}})$ is of measure zero as well. By (8), we have that $\Phi(B)$ is of measure zero. Then, from (7), Lemmas 17 and 18, we get the following proposition.

Proposition 19 *There exists an open and full measure subset Θ of Λ such that for each $(\xi, e, a) \in \widetilde{F}^{-1}(0) \cap (\Xi \times \Theta)$ and for each $h \in \mathcal{H}$, either*

$$\mu_h > 0 \text{ or } \chi_h(x_h, e_h) + a_h > 0$$

Finally, observe that the above proposition implies that $\widetilde{F}$ is differentiable in

$\widetilde{F}^{-1}(0) \cap (\Xi \times \Theta)$. Indeed, given $(\xi^*, e^*, a^*) \in \widetilde{F}^{-1}(0) \cap (\Xi \times \Theta)$, define

$$\begin{aligned}\mathcal{H}_1(\xi^*, e^*, a^*) &:= \mathcal{H}_1^* := \{h \in \mathcal{H} : \mu_h^* = 0 \text{ and } \chi_h(x_h^*, e_h^*) + a_h^* > 0\} \\ \mathcal{H}_2(\xi^*, e^*, a^*) &:= \mathcal{H}_2^* := \{h \in \mathcal{H} : \mu_h^* > 0 \text{ and } \chi_h(x_h^*, e_h^*) + a_h^* = 0\}\end{aligned} \tag{10}$$

By Proposition 19, we get $\mathcal{H}_1^* \cup \mathcal{H}_2^* = \mathcal{H}$ and $\mathcal{H}_1^* \cap \mathcal{H}_2^* = \emptyset$. Since linear and possibility functions are continuous, there is an open neighborhood I^* of (ξ^*, e^*, a^*) in $\Xi \times \Theta$ such that for each $(\xi, e, a) \in I^*$,

$$\widetilde{F}^{h.3}(\xi, e, a) = \begin{cases} \mu_h \text{ if } h \in \mathcal{H}_1^* \\ \chi_h(x_h, e_h) + a_h \text{ if } h \in \mathcal{H}_2^* \end{cases}$$

5.2 Regularity in an open and full measure subset

In this subsection, we prove Theorem 13. The proof of Theorem 13 is built upon Proposition 21 which is the main result of this subsection. From now on, the set Θ is the open and full measure subset of Λ obtained in Proposition 19, and the domain of $\widetilde{F}$ given in (6) will be $\Xi \times \Theta$ instead of $\Xi \times \Lambda$. To prove Proposition 21, we need the following definitions and the following key lemma, namely Lemma 20.

Observe that there is a slight difference between the below definitions and the ones given in Subsection 5.1. In Subsection 5.1, we considered the set $\mathcal{P}$ of appropriate $\{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\}$ with $\mathcal{H}_3 \neq \emptyset$. In this subsection, we are interested to describe the case in which $\mathcal{H}_3 = \emptyset$. Then, we define

$$\mathcal{A} := \{\mathcal{I} := \{\mathcal{H}_1, \mathcal{H}_2\} : \ \mathcal{H}_i \subseteq \mathcal{H}, \ \forall i = 1, 2; \mathcal{H}_1 \cup \mathcal{H}_2 = \mathcal{H} \text{ and } \mathcal{H}_1 \cap \mathcal{H}_2 = \emptyset\}$$

Let $\mathcal{I} = \{\mathcal{H}_1, \mathcal{H}_2\} \in \mathcal{A}$, for each $i = 1, 2$ denote by $\mathcal{H}_i(\mathcal{I})$ the set $\mathcal{H}_i$ in $\mathcal{I}$, and by $|\mathcal{H}_i(\mathcal{I})|$ the number of element of $\mathcal{H}_i(\mathcal{I})$. Define

$$\Xi_{\mathcal{I}} := \mathbb{R}_{++}^{(C+1)H} \times (\mathbb{R}^{|\mathcal{H}_1(\mathcal{I})|} \times \mathbb{R}_{++}^{|\mathcal{H}_2(\mathcal{I})|}) \times \mathbb{R}_{++}^{(C-1)}$$

Observe that $\dim \Xi_{\mathcal{I}} = \dim \Xi$. Define the function $\widetilde{F}_{\mathcal{I}} : \Xi_{\mathcal{I}} \times \Theta \to \mathbb{R}^{\dim \Xi_{\mathcal{I}}}$

$$\widetilde{F}_{\mathcal{I}}(\xi, e, a) := ((\widetilde{F}^{h.1}(\xi, e, a), \widetilde{F}^{h.2}(\xi, e, a), \widetilde{F}^{h.3}(\xi, e, a))_{h \in \mathcal{H}}, \widetilde{F}^M(\xi, e, a)) \tag{11}$$

where $\widetilde{F}_{\mathcal{I}}$ differs from $\widetilde{F}$ defined in (6), for the domain and for the component $\widetilde{F}_{\mathcal{I}}^{h.3}$ defined below

$$\widetilde{F}_{\mathcal{I}}^{h.3}(\xi, e, a) := \begin{cases} \mu_h \text{ if } h \in \mathcal{H}_1(\mathcal{I}), \\ \chi_h(x_h, e_h) + a_h \text{ if } h \in \mathcal{H}_2(\mathcal{I}) \end{cases}$$

All the above definitions allows us to conclude that for each $\mathcal{I} \in \mathcal{A}$, $\widetilde{F}_{\mathcal{I}}$ is differentiable on all its domain.

Lemma 20 *For each $\mathcal{I} \in \mathcal{A}$ and for each $(\xi^*, e^*, a^*) \in \widetilde{F}_{\mathcal{I}}^{-1}(0)$, the Jacobian matrix $D_{(\xi,e,a)}\widetilde{F}_{\mathcal{I}}(\xi^*, e^*, a^*)$ has full row rank.*

Observe that by (10), for each $(\xi^*, e^*, a^*) \in \widetilde{F}^{-1}(0)$ we get

(1) $\mathcal{I}^* := \{\mathcal{H}_1^*, \mathcal{H}_2^*\} \in \mathcal{A}$,
(2) $(\xi^*, e^*, a^*) \in \widetilde{F}_{\mathcal{I}^*}^{-1}(0)$, and
(3) $D_{(\xi,e,a)}\widetilde{F}(\xi^*, e^*, a^*) = D_{(\xi,e,a)}\widetilde{F}_{\mathcal{I}^*}(\xi^*, e^*, a^*)$.

Then, from Lemma 20 we can state the following result.

Proposition 21 0 *is a regular value for $\widetilde{F}$.*

From the above proposition and Sard's Theorem (see Theorem 32 in Appendix B), there is a full measure subset Θ^* of Θ such that for each $(e, a) \in \Theta^*$, 0 is a regular value for $\widetilde{F}_{e,a}$. Since Θ is a full measure subset of Λ, Θ^* is a full measure subset of Λ. Since the following set

$$\Lambda^r := \{(e, a) \in \Lambda : \mathcal{E}^a = (e, \chi^a, u) \text{ is regular}\}$$

contains Θ^*, Λ^r is a full measure subset of Λ. Moreover, from Lemma 17 and Corollary 34 in Appendix B, it follows that Λ^r is an open subset of Λ. Therefore, Theorem 13 holds true.

5.3 Generic regularity

In this subsection, we take for fixed an arbitrary $u \in \mathcal{U}$ and we prove Corollary 14 using several lemmas, namely Lemmas 22, 23, 24, 25 and 26.[18]

Observe that the global equilibrium function F defined in (5) is a continuous function. Indeed, the convergence of a sequence $(\chi_h^v)_{v\in\mathbb{N}}$ means uniform convergence on compacta of $(\chi_h^v)_{v\in\mathbb{N}}$ and of $(D\chi_h^v)_{v\in\mathbb{N}}$ (see Appendix B).

By the following lemma, we deduce that the set $\mathcal{R}$ of (e, χ) such that $\mathcal{E} = (e, \chi, u)$ is a regular economy is a dense subset of $\mathbb{R}_{++}^{CH} \times \mathcal{X}$.

Lemma 22 *There exists a dense subset D of $\mathbb{R}_{++}^{CH} \times \mathcal{X}$ such that for each $(e, \chi) \in D$ the economy $\mathcal{E} = (e, \chi, u)$ is regular.*

[18] To show Corollary 14, we follow a similar strategy to the one presented by Citanna, Kajii and Villanacci (1998).

To prove that $\mathcal{R}$ is open we need introduce some preliminary definitions and lemmas. Define the restriction to $F^{-1}(0)$ of the projection of $\Xi \times \mathbb{R}^{CH}_{++} \times \mathcal{X}$ onto $\mathbb{R}^{CH}_{++} \times \mathcal{X}$,

$$\Pi : (\xi, e, \chi) \in F^{-1}(0) \to \Pi(\xi, e, \chi) := (e, \chi) \in \mathbb{R}^{CH}_{++} \times \mathcal{X}$$

Theorem 7 and Definition 8 imply that $\mathcal{R} \subseteq (\mathbb{R}^{CH}_{++} \times \mathcal{X}) \setminus \Pi(C_1)$ where

$$C_1 := \{(\xi^*, e, \chi) \in F^{-1}(0) : F_{\mathcal{E}} \text{ is not differentiable at } \xi^*\}$$

The closedness of $\Pi(C_1)$ in $\mathbb{R}^{CH}_{++} \times \mathcal{X}$ follows from the closedness of C_1 in $F^{-1}(0)$ and the properness of Π obtained by the following two lemmas. [19]

Lemma 23 *The set C_1 is closed in $F^{-1}(0)$.*

Lemma 24 *The function Π is proper.*

Then, the set $A := (\mathbb{R}^{CH}_{++} \times \mathcal{X}) \setminus \Pi(C_1)$ is open in $\mathbb{R}^{CH}_{++} \times \mathcal{X}$. From now on, the domain of F will be $\Xi \times A$ instead of $\Xi \times \mathbb{R}^{CH}_{++} \times \mathcal{X}$. Define also the restriction to $F^{-1}(0)$ of the projection of $\Xi \times A$ onto A,

$$\Psi : (\xi, e, \chi) \in F^{-1}(0) \to \Psi(\xi, e, \chi) := (e, \chi) \in A$$

Theorem 7 and Definition 8 imply that $\mathcal{R} = A \setminus \Psi(C_2)$ where

$$C_2 := \{(\xi^*, e, \chi) \in F^{-1}(0) : \operatorname{rank} D_\xi F_{\mathcal{E}}(\xi^*) < \dim \Xi\}$$

The closedness of $\Psi(C_2)$ in A follows from the closedness of C_2 in $F^{-1}(0)$ and the properness of Ψ obtained by the following two lemmas. Then, $\mathcal{R}$ is open in A. Finally, $\mathcal{R}$ is open in $\mathbb{R}^{CH}_{++} \times \mathcal{X}$ since A is open in $\mathbb{R}^{CH}_{++} \times \mathcal{X}$.

Lemma 25 *The set C_2 is closed in $F^{-1}(0)$.*

Lemma 26 *The function Ψ is proper.*

Appendix A

In this appendix we show all the results stated in Sections 4 and 5. Moreover, to prove some of these results we present the following two propositions, namely Propositions 27 and 29.

As a consequence of points 1-4 of Assumption 2 we get Proposition 27. It plays a fundamental role in the characterization of household h's maximization problem in terms of Kuhn–Tucker conditions (see Proposition 5 and its proof)

[19] Also see the definition of proper function, i.e., Definition 33 in Appendix B.

and in the result of generic regularity (in particular see the proof of Lemmas 18 and 20).

Proposition 27 *Let $\mathcal{E}$ be an economy satisfying Assumption 2. If $(x_h, p) \in \mathbb{R}^C_{++} \times \mathbb{R}^C_{++}$ is such that $\chi_h(x_h, e_h) = 0$ and $p(x_h - e_h) = 0$, then p and $D_{x_h}\chi_h(x_h, e_h)$ are linearly independent.*

Proof. Otherwise, suppose that $D_{x_h}\chi_h(x_h, e_h) = \beta p$ with $\beta \neq 0$. Since $p \gg 0$ and $\chi_h(x_h, e_h) = 0$, by point 4 of Assumption 2, we get $\beta > 0$. Then, $D_{x_h}\chi_h(x_h, e_h) \gg 0$. By point 3 of Assumption 2, $\widetilde{x}_h \in \mathbb{R}^C_{++}$ satisfies $\chi_h(\widetilde{x}_h, e_h) > 0$ and $\widetilde{x}_h \ll e_h$. From points 1 and 2 of Assumption 2, χ_h is C^1 and quasi-concave, then $\chi_h(\widetilde{x}_h, e_h) - \chi_h(x_h, e_h) > 0$ implies $D_{x_h}\chi_h(x_h, e_h)(\widetilde{x}_h - x_h) \geq 0$. Therefore, $D_{x_h}\chi_h(x_h, e_h)(e_h - x_h) > 0$ since $D_{x_h}\chi_h(x_h, e_h) \gg 0$ and $\widetilde{x}_h \ll e_h$. That is, $\beta p(e_h - x_h) > 0$ which contradicts $p(x_h - e_h) = 0$. ■

In the following remark we just observe that at the solution of household h's maximization problem an analogous condition to Smale's Assumption holds true (see *NCP Hypothesis* in Smale, 1974b).

Remark 28 *Let $\mathcal{E}$ be an economy satisfying Assumptions 1 and 2, and x^*_h be the solution to problem (1) at $\mathcal{E}$ and $p^{*\setminus} \in \mathbb{R}^{C-1}_{++}$. From Propositions 5 and 27, point 1 of Assumption 1 and point 4 of Assumption 2, we have that $\chi_h(x^*_h, e_h) = 0$ implies that $D_{x_h}u_h(x^*_h)$ and $D_{x_h}\chi_h(x^*_h, e_h)$ are linearly independent.*

As a direct consequence of Remark 12, points 1 and 2 of Assumption 2, we obtain the following proposition. The continuous selection functions given in Proposition 29 play a fundamental role in the *properness* result used to show differentiability and regularity results (see Lemma 17 and its proof).

Proposition 29 *Let $h \in \mathcal{H}$, Λ_h denotes the projection of Λ onto $\mathbb{R}^C_{++} \times \mathbb{R}$. For each $h \in \mathcal{H}$, there exists a continuous function $\widetilde{x}_h : \Lambda_h \to \mathbb{R}^C_{++}$ such that for each $(e_h, a_h) \in \Lambda_h$, $\chi_h(\widetilde{x}_h(e_h, a_h), e_h) + a_h > 0$ and $\widetilde{x}_h(e_h, a_h) \ll e_h$.*

Proof. Let $h \in \mathcal{H}$. By Remark 12, for each $(e_h, a_h) \in \Lambda_h$ the following set $\left\{x_h \in \mathbb{R}^C_{++} : \chi_h(x_h, e_h) + a_h > 0 \text{ and } x_h \ll e_h\right\}$ is not empty, and by point 2 of Assumption 2 it is a convex set. Define the correspondence $\phi_h : \Lambda_h \rightrightarrows \mathbb{R}^C$

$$\phi_h : (e_h, a_h) \rightrightarrows \phi_h(e_h, a_h) := \left\{x_h \in \mathbb{R}^C_{++} : \chi_h(x_h, e_h) + a_h > 0 \text{ and } x_h \ll e_h\right\}$$

From point 1 of Assumption 2, for each $x_h \in \mathbb{R}^C$ the following set

$$\phi_h^{-1}(x_h) := \{(e_h, a_h) \in \Lambda_h : \chi_h(x_h, e_h) + a_h > 0 \text{ and } x_h \ll e_h\}$$

is open in Λ_h. Moreover, Λ_h equipped with the metric induced by the Euclidean distance is metrizable, thus paracompact. Then, we have the desired result since the correspondence ϕ_h satisfies the assumptions of Michael's Selection theorem (see Proposition 1.5.1 in Florenzano, 2003). ■

Proof of Proposition 5. The existence of a solution to problem (1) follows from points 1 and 4 of Assumption 1, and points 1 and 3 of Assumption 2. The uniqueness follows from point 3 of Assumptions 1 and point 2 of Assumption 2. By the well known Kuhn–Tucker theorem for non-linear programming, the sufficiency of the Kuhn–Tucker conditions follows from point 3 of Assumption 1 and point 2 of Assumption 2. The necessity of the Kuhn–Tucker conditions follows from Propositions 27. Finally, from point 2 of Assumption 1 and point 4 of Assumption 2, we get $\lambda_h^* \in \mathbb{R}_{++}$. ■

Proof of Theorem 7. Theorem 7 is a particular case of existence and compactness results obtained in del Mercato (2006a) using homotopy arguments. Then, we just provide the homotopy H adapted to our model and we return the reader to del Mercato (2006a) for the detailed proof.[20] Take for fixed $\mathcal{E}$ and define the following homotopy $H : \Xi \times [0,1] \to \mathbb{R}^{\dim \Xi}$,

$$H(\xi, t) := \begin{cases} \Psi(\xi, 1-2t) \text{ if } 0 \leq t \leq \frac{1}{2} \\ \Gamma(\xi, 2-2t) \text{ if } \frac{1}{2} < t \leq 1 \end{cases}$$

where $\Psi, \Gamma : \Xi \times [0,1] \to \mathbb{R}^{\dim \Xi}$ are defined by

$$\Psi(\xi,\tau) := ((\Psi^{h.1}(\xi,\tau), \Psi^{h.2}(\xi,\tau), \Psi^{h.3}(\xi,\tau))_{h\in\mathcal{H}}, \Psi^M(\xi,\tau))$$

$$\Psi^{h.1}(\xi,\tau) = D_{x_h}u_h(x_h) - \lambda_h p,\ \Psi^{h.2}(\xi,\tau) = -p(x_h - e_h^\tau),$$

$$\Psi^{h.3}(\xi,\tau) = \min\{\mu_h, \chi_h(\widetilde{x}_h, e_h)\} \text{ and } \Psi^M(\xi,\tau) = \sum_{h\in\mathcal{H}} x_h^{\setminus} - \sum_{h\in\mathcal{H}} e_h^{\tau\setminus},$$

and

$$\Gamma(\xi,\tau) := ((\Gamma^{h.1}(\xi,\tau), \Gamma^{h.2}(\xi,\tau), \Gamma^{h.3}(\xi,\tau))_{h\in\mathcal{H}}, \Gamma^M(\xi,\tau))$$

$$\Gamma^{h.1}(\xi,\tau) = D_{x_h}u_h(x_h) - \lambda_h p + \mu_h(1-\tau)D_{x_h}\chi_h((1-\tau)x_h + \tau\widetilde{x}_h, e_h),$$

$$\Gamma^{h.2}(\xi,\tau) = -p(x_h - e_h),\ \Gamma^{h.3}(\xi,\tau) = \min\{\mu_h, \chi_h((1-\tau)x_h + \tau\widetilde{x}_h, e_h)\}$$

$$\text{and } \Gamma^M(\xi,\tau) = \sum_{h\in\mathcal{H}} x_h^{\setminus} - \sum_{h\in\mathcal{H}} e_h^{\setminus}.$$

For each $h \in \mathcal{H}$, $\widetilde{x}_h$ is given by point 3 of Assumption 2, and for each $\tau \in [0,1]$, $e_h^\tau := (1-\tau)e_h + \tau x_h^{**}$, where $(x_h^{**})_{h\in\mathcal{H}}$ is a Pareto optimal allocation in the exchange economy à la Debreu $\mathcal{E}_D := (X_h, u_h, e_h)_{h\in\mathcal{H}}$ where $X_h := \mathbb{R}_{++}^C$ for every $h \in \mathcal{H}$. ■

[20] Observe that in del Mercato (2006a), utility and possibility functions also depend on the consumptions of all households.

Proof of Proposition 16. It is enough to adapt the proof of Theorem 7. Take for fixed $(e,a) \in \Lambda$, the homotopy $H : \Xi \times [0,1] \to \mathbb{R}^{\dim \Xi}$ does not change. The homotopies $\Psi, \Gamma : \Xi \times [0,1] \to \mathbb{R}^{\dim \Xi}$ adapted to this case differ from the ones used in the proof of Theorem 7 since $\Psi^{h.3}(\xi,\tau) = \min\{\mu_h, \chi_h(\widetilde{x}_h, e_h) + a_h\}$ and $\Gamma^{h.3}(\xi,\tau) = \min\{\mu_h, \chi_h((1-\tau)x_h + \tau\widetilde{x}_h, e_h) + a_h\}$, where for each $h \in \mathcal{H}$, $\widetilde{x}_h$ is given by the definition of Λ (see Definition 11). ■

Proof of Lemma 17. We have to show that any sequence $(\xi^v, e^v, a^v)_{v\in\mathbb{N}} \subseteq \widetilde{F}^{-1}(0)$, up to a subsequence, converges to an element of $\widetilde{F}^{-1}(0)$, knowing that $(e^v, a^v)_{v\in\mathbb{N}} \in \Lambda$, up to a subsequence, converges to $(e^*, a^*) \in \Lambda$.

• $(x^v)_{v\in\mathbb{N}}$, *up to a subsequence, converges to* $x^* \in \mathbb{R}^{CH}_{++}$.

$(x^v)_{v\in\mathbb{N}} \subseteq \mathbb{R}^{CH}_{++}$, and from $\widetilde{F}^M(\xi^v, e^v, a^v) = 0$ and $\widetilde{F}^{k.2}(\xi^v, e^v, a^v) = 0$ in (6), $x^v_k = \sum_{h\in\mathcal{H}} e^v_h - \sum_{h\neq k} x^v_h \leq \sum_{h\in\mathcal{H}} e^v_h$ for each $k \in \mathcal{H}$. Since $(e^v_h)_{v\in\mathbb{N}}$ converges to $e^*_h \in \mathbb{R}^C_{++}$ for each $h \in \mathcal{H}$, $(x^v)_{v\in\mathbb{N}}$ is bounded from above by an element of $\mathbb{R}^C_{++}$. Then, $(x^v)_{v\in\mathbb{N}}$, up to a subsequence, converges to $x^* \geq 0$.

Now, we prove that $x^*_h \gg 0$ for each $h \in \mathcal{H}$. By $\widetilde{F}^{h.1}(\xi^v, e^v, a^v) = 0$ and $\widetilde{F}^{h.2}(\xi^v, e^v, a^v) = 0$ in (6), $u_h(x^v_h) \geq u_h(\widetilde{x}_h(e^v_h, a^v_h))$ for every $v \in \mathbb{N}$, where $\widetilde{x}_h$ is the continuous selection function given by Proposition 29. Define $\mathbf{1} := (1, ..., 1) \in \mathbb{R}^C_{++}$, from point 2 of Assumption 1 we have that for each $\varepsilon > 0$, $u_h(x^v_h + \varepsilon\mathbf{1}) \geq u_h(\widetilde{x}_h(e^v_h, a^v_h))$ for every $v \in \mathbb{N}$. So taking the limit on v, since $(e^v_h, a^v_h)_{v\in\mathbb{N}}$ converges to $(e^*_h, a^*_h) \in \Lambda_h$, and u_h and $\widetilde{x}_h$ are continuous, then we get $u_h(x^*_h + \varepsilon\mathbf{1}) \geq u_h(\widetilde{x}_h(e^*_h, a^*_h)) := \underline{u}_h$ for each $\varepsilon > 0$. By point 4 of Assumption 1, $x^*_h \in \mathbb{R}^C_{++}$ since x^*_h belongs to the set $cl_{\mathbb{R}^C}\{x_h \in \mathbb{R}^C_{++} : u_h(x_h) \geq \underline{u}_h\}$.

• $(\lambda^v, \mu^v)_{v\in\mathbb{N}}$, *up to a subsequence, converges to* $(\lambda^*, \mu^*) \in \mathbb{R}^H_+ \times \mathbb{R}^H_+$.

It is enough to show that $(\lambda^v_h p^v, \mu^v_h)_{v\in\mathbb{N}}$ is bounded for each $h \in \mathcal{H}$. Then, $(\lambda^v_h p^v, \mu^v_h)_{v\in\mathbb{N}} \subseteq \mathbb{R}^C_{++} \times \mathbb{R}_+$, up to a subsequence, converges to $(\pi^*_h, \mu^*_h) \in \mathbb{R}^C_+ \times \mathbb{R}_+$, and $\lambda^*_h = \pi^{*C}_h$ since $p^{vC} = 1$ for each $v \in \mathbb{N}$.

Suppose otherwise that there is a subsequence of $(\lambda^v_h p^v, \mu^v_h)_{v\in\mathbb{N}}$ (that without loss of generality we continue to denote with $(\lambda^v_h p^v, \mu^v_h)_{v\in\mathbb{N}}$) such that $\|(\lambda^v_h p^v, \mu^v_h)\| \longrightarrow +\infty$. Consider the sequence $\left(\frac{\lambda^v_h p^v, \mu^v_h}{\|(\lambda^v_h p^v, \mu^v_h)\|}\right)_{v\in\mathbb{N}}$ in the sphere, a compact set. Then, up to a subsequence $\left(\frac{\lambda^v_h p^v, \mu^v_h}{\|(\lambda^v_h p^v, \mu^v_h)\|}\right) \longrightarrow (\pi_h, \mu_h) \neq 0$. Since $\mu^v_h \geq 0$ and $\lambda^v_h p^v \gg 0$ for each $v \in \mathbb{N}$, we get $\pi_h \geq 0$ and $\mu_h \geq 0$. By $\widetilde{F}^{h.1}(\xi^v, e^v, a^v) = 0$ in (6), $\lambda^v_h p^v = D_{x_h} u_h(x^v_h) + \mu^v_h D_{x_h}\chi_h(x^v_h, e^v_h)$ for each $v \in \mathbb{N}$. Now, divide both sides by $\|(\lambda^v_h p^v, \mu^v_h)\|$ and take the limits. From point 1 of Assumption 1 and point 1 of Assumption 2, we get

$$\pi_h = \mu_h D_{x_h}\chi_h(x^*_h, e^*_h)$$

$\mu_h > 0$, otherwise we get $(\pi_h, \mu_h) = 0$. By point 4 of Assumption 2, we have $D_{x_h}\chi_h(x_h^*, e_h^*) \neq 0$. Then, $\pi_h \neq 0$. From Kuhn-Tucker necessary and sufficient conditions, we get

$$\begin{aligned} \pi_h x_h^* = \min_{x_h \in \mathbb{R}^C_{++}} \quad & \pi_h x_h \\ \text{subject to } & \chi_h(x_h, e_h^*) \geq 0 \end{aligned} \tag{12}$$

By $\widetilde{F}^{h.2}(\xi^v, e^v, a^v) = 0$ in (6), we get $\lambda_h^v p^v x_h^v = \lambda_h^v p^v e_h^v$ for each $v \in \mathbb{N}$. Now, divide both sides by $\|(\lambda_h^v p^v, \mu_h^v)\|$ and take the limits. We get $\pi_h x_h^* = \pi_h e_h^*$. By point 3 of Assumption 2, $\chi_h(\widetilde{x}_h, e_h^*) > 0$ and $\pi_h \widetilde{x}_h < \pi_h e_h^* = \pi_h x_h^*$ contradict (12).

- *$(p^{v\backslash})_{v \in \mathbb{N}}$, up to a subsequence, converges to $p^{*\backslash} \in \mathbb{R}^{C-1}_{++}$.*

By point 2 of Assumption 1 and point 5 of Assumption 2, $\lambda_k^* = D_{x_k}u_k(x_k^*) + \mu_k^* D_{x_k}\chi_k(x_k^*, e_k^*) > 0$, for some $k = h(C) \in \mathcal{H}$. From the previous step, $(\lambda_k^v p^{v\backslash})_{v \in \mathbb{N}}$ admits a subsequence converging to $\pi_k^{*\backslash} \geq 0$. Then, $(p^{v\backslash})_{v \in \mathbb{N}}$, up to a subsequence, converges to $p^{*\backslash} \geq 0$, since $\lambda_k^* > 0$. Now, suppose that there is $c \neq C$, such that $p^{*c} = 0$. By point 2 of Assumption 1 and point 5 of Assumption 2, for some $k' = h(c) \in \mathcal{H}$ we get $0 < D_{x_{k'}^c}u_{k'}(x_{k'}^*) + \mu_{k'}^* D_{x_{k'}^c}\chi_{k'}(x_{k'}^* e_{k'}^*) = \lambda_{k'}^* p^{*c} = 0$ that is a contradiction.

- $\lambda^* \in \mathbb{R}^H_{++}$.

Otherwise, suppose that $\lambda_h^* = 0$ for some $h \in \mathcal{H}$. By $\widetilde{F}^{h.1}(\xi^v, e^v, a^v) = 0$ in (6), $\lambda_h^v p^v = D_{x_h}u_h(x_h^v) + \mu_h^v D_{x_h}\chi_h(x_h^v, e_h^v)$ for each $v \in \mathbb{N}$. Taking the limit, from point 1 of Assumptions 1 and 2 we get $0 = \lambda_h^* p^* = D_{x_h}u_h(x_h^*) + \mu_h^* D_{x_h}\chi_h(x_h^*, e_h^*)$. By point 2 of Assumption 1 and point 4 of Assumption 2, for some good $c = 1, .., C$ we get $0 < D_{x_h^c}u_h(x_h^*) + \mu_h^* D_{x_h^c}\chi_h(x_h^*, e_h^*) = \lambda_h^* p^{*c} = 0$ that is a contradiction. ■

Proof of Lemma 18. We have to show that for each $(\xi^*, e^*, a^*) \in \widetilde{F}^{-1}_{\mathcal{J},\bar{h}}(0)$, the Jacobian matrix $D_{(\xi,e,a)}\widetilde{F}_{\mathcal{J},\bar{h}}(\xi^*, e^*, a^*)$ has full row rank.

Let $\Delta := ((\Delta x_h, \Delta\lambda_h, \Delta\mu_h)_{h \in \mathcal{H}}, \Delta p^\backslash, \Delta v) \in \mathbb{R}^{(C+2)H} \times \mathbb{R}^{C-1} \times \mathbb{R}$. It is enough to show that $\Delta \cdot D_{(\xi,e,a)}\widetilde{F}_{\mathcal{J},\bar{h}}(\xi^*, e^*, a^*) = 0$ implies $\Delta = 0$. To prove it, we consider the computation of the partial Jacobian matrix with respect to the following variables

$$((x_h, \lambda_h, \mu_h)_{h \in \mathcal{H}}, e_{\bar{h}}, a_{\bar{h}})$$

The partial system $\Delta \cdot D_{(\xi,e,a)}\widetilde{F}_{\mathcal{J},\bar{h}}(\xi^*, e^*, a^*) = 0$ is written in detail below.

Without loss of generality, we denote $\mathcal{H}_i := \mathcal{H}_i(\mathcal{J})$ for each $i = 1, 2, 3$.

$$\begin{cases} \Delta x_h \cdot D^2_{x_h} u_h(x^*_h) - \Delta\lambda_h p^* + \Delta p^\setminus \cdot [I_{C-1}|0] = 0 \text{ if } h \in \mathcal{H}_1 \cup (\mathcal{H}_3 \setminus \{\bar{h}\}) \\ \Delta x_{\bar{h}} \cdot D^2_{x_{\bar{h}}} u_{\bar{h}}(x^*_{\bar{h}}) - \Delta\lambda_{\bar{h}} p^* + \Delta p^\setminus \cdot [I_{C-1}|0] + \Delta v D_{x_{\bar{h}}} \chi_{\bar{h}}(x^*_{\bar{h}}, e^*_{\bar{h}}) = 0 \\ \Delta x_{h'} \cdot \left(D^2_{x_{h'}} u_{h'}(x^*_{h'}) + \mu^*_{h'} D^2_{x_{h'}} \chi_{h'}(x^*_{h'}, e^*_{h'}) \right) - \Delta\lambda_{h'} p^* + \\ \Delta\mu_{h'} D_{x_{h'}} \chi_{h'}(x^*_{h'}, e^*_{h'}) + \Delta p^\setminus \cdot [I_{C-1}|0] = 0 \text{ if } h' \in \mathcal{H}_2 \\ -\Delta x_h p^* = 0 \text{ for each } h \in \mathcal{H} \\ \Delta x_h D_{x_h} \chi_h(x^*_h, e^*_h) + \Delta\mu_h = 0 \text{ if } h \in \mathcal{H}_1 \cup \mathcal{H}_3 \\ \Delta x_{h'} D_{x_{h'}} \chi_{h'}(x^*_{h'}, e^*_{h'}) = 0 \text{ if } h' \in \mathcal{H}_2 \\ \Delta\lambda_{\bar{h}} p^* - \Delta p^\setminus \cdot [I_{C-1}|0] + \Delta v D_{e_{\bar{h}}} \chi_{\bar{h}}(x^*_{\bar{h}}, e^*_{\bar{h}}) = 0 \\ \Delta v = 0 \end{cases} \tag{13}$$

Since $\Delta v = 0$ and $p^{*C} = 1$, then $\Delta\lambda_{\bar{h}} = 0$ and $\Delta p^\setminus = 0$. From the above system , we get

$$\left(\Delta x_h \cdot D^2_{x_h} u_h(x^*_h) \right) \Delta x_h = 0 \text{ if } h \in \mathcal{H}_1 \cup \mathcal{H}_3 \tag{14}$$

and

$$\left(\Delta x_{h'} \cdot D^2_{x_{h'}} u_{h'}(x^*_{h'}) \right) \Delta x_{h'} = -\mu^*_{h'} \left(\Delta x_{h'} \cdot D^2_{x_{h'}} \chi_{h'}(x^*_{h'}, e^*_{h'}) \right) \Delta x_{h'} \text{ if } h' \in \mathcal{H}_2$$

Point 2 of Assumption 2 and $\mu^*_{h'} > 0$ for each $h' \in \mathcal{H}_2$ imply that

$$\left(\Delta x_{h'} \cdot D^2_{x_{h'}} u_{h'}(x^*_{h'}) \right) \Delta x_{h'} \geq 0 \text{ if } h' \in \mathcal{H}_2 \tag{15}$$

Observe that from $\widetilde{F}^{\bar{h}.1}_{\mathcal{J},\bar{h}}(\xi^*, e^*, a^*) = 0$ in (9) and system (13), we get

$$D_{x_h} u_h(x^*_h) \Delta x_h = \lambda^*_h p^* \Delta x_h = 0 \text{ if } h \in \mathcal{H}_1 \cup \mathcal{H}_3, \text{ and}$$

$$D_{x_{h'}} u_{h'}(x^*_{h'}) \Delta x_{h'} = \lambda^*_{h'} p^* \Delta x_{h'} - \mu^*_{h'} D_{x_{h'}} \chi_{h'}(x^*_{h'}, e^*_{h'}) \Delta x_{h'} = 0 \text{ if } h' \in \mathcal{H}_2$$

Then, (14), (15) and point 3 of Assumption 1 imply that $\Delta x_h = 0$ for each $h \in \mathcal{H}$. Therefore, the relevant equations of system (13) become

$$\begin{cases} \Delta\lambda_h p^* = 0 \text{ if } h \in \mathcal{H}_1 \cup (\mathcal{H}_3 \setminus \{\bar{h}\}) \\ \Delta\lambda_{h'} p^* - \Delta\mu_{h'} D_{x_{h'}} \chi_{h'}(x^*_{h'}, e^*_{h'}) = 0 \text{ if } h' \in \mathcal{H}_2 \\ \Delta\mu_h = 0 \text{ if } h \in \mathcal{H}_1 \cup \mathcal{H}_3 \end{cases}$$

$\Delta\lambda_h = 0$ for each $h \in \mathcal{H}_1 \cup (\mathcal{H}_3 \setminus \{\bar{h}\})$, since $p^* \gg 0$. From $\widetilde{F}_{\mathcal{J},\bar{h}}(\xi^*, e^*, a^*) = 0$ in (9) and Proposition 27, we have that p^* and $D_{x_{h'}} \chi_{h'}(x^*_{h'}, e^*_{h'})$ are linearly independent for each $h' \in \mathcal{H}_2$. Then, $\Delta\lambda_{h'} = \Delta\mu_{h'} = 0$ for each $h' \in \mathcal{H}_2$. Therefore, $\Delta = 0$. ■

Proof of Lemma 20. Let $\Delta := ((\Delta x_h, \Delta\lambda_h, \Delta\mu_h)_{h\in\mathcal{H}}, \Delta p^{\setminus}) \in \mathbb{R}^{(C+2)H} \times \mathbb{R}^{C-1}$. It is enough to show that $\Delta \cdot D_{(\xi,e,a)}\widetilde{F}_{\mathcal{I}}(\xi^*, e^*, a^*) = 0$ implies $\Delta = 0$. We consider two cases: 1. $\mathcal{H}_1(\mathcal{I}) \neq \emptyset$, and 2. $\mathcal{H}_1(\mathcal{I}) = \emptyset$. Without loss of generality, we denote $\mathcal{H}_i := \mathcal{H}_i(\mathcal{I})$ for each $i = 1, 2$.

Case 1. $\mathcal{H}_1 \neq \emptyset$. Without loss of generality we suppose $1 \in \mathcal{H}_1$. In this case, we consider the computation of the partial Jacobian matrix with respect to the following variables

$$((x_h, \lambda_h, \mu_h)_{h\in\mathcal{H}}, e_1)$$

The partial system $\Delta \cdot D_{(\xi,e,a)}\widetilde{F}_{\mathcal{I}}(\xi^*, e^*, a^*) = 0$ is written in detail below.

$$\begin{cases} \Delta x_h \cdot D^2_{x_h} u_h(x^*_h) - \Delta\lambda_h p^* + \Delta p^{\setminus} \cdot [I_{C-1}|0] = 0 \text{ if } h \in \mathcal{H}_1 \\ \Delta x_{h'} \cdot \Big(D^2_{x_{h'}} u_{h'}(x^*_{h'}) + \mu^*_{h'} D^2_{x_{h'}} \chi_{h'}(x^*_{h'}, e^*_{h'})\Big) - \Delta\lambda_{h'} p^* + \\ \Delta\mu_{h'} D_{x_{h'}} \chi_{h'}(x^*_{h'}, e^*_{h'}) + \Delta p^{\setminus} \cdot [I_{C-1}|0] = 0 \text{ if } h' \in \mathcal{H}_2 \\ -\Delta x_h p^* = 0 \text{ for each } h \in \mathcal{H} \\ \Delta x_h D_{x_h} \chi_h(x^*_h, e^*_h) + \Delta\mu_h = 0 \text{ if } h \in \mathcal{H}_1 \\ \Delta x_{h'} D_{x_{h'}} \chi_{h'}(x^*_{h'}, e^*_{h'}) = 0 \text{ if } h' \in \mathcal{H}_2 \\ \Delta\lambda_1 p^* - \Delta p^{\setminus} \cdot [I_{C-1}|0] = 0 \end{cases}$$

Since $p^{*C} = 1$, we get $\Delta\lambda_1 = 0$ and $\Delta p^{\setminus} = 0$. From the above system, using similar arguments as in the proof of Lemma 18, we get $\Delta x_h = 0$ for each $h \in \mathcal{H}$. Then, $\Delta\mu_h = 0$ for each $h \in \mathcal{H}_1$, and $\Delta\lambda_h = 0$ for each $h \in \mathcal{H}_1 \setminus \{1\}$ since $p^* \gg 0$. From $\widetilde{F}_{\mathcal{I}}(\xi^*, e^*, a^*) = 0$ in (11) and Proposition 27, we have that p^* and $D_{x_{h'}} \chi_{h'}(x^*_{h'}, e^*_{h'})$ are linearly independent for each $h' \in \mathcal{H}_2$. Then, $\Delta\lambda_{h'} = \Delta\mu_{h'} = 0$ for each $h' \in \mathcal{H}_2$. Therefore, $\Delta = 0$.

Case 2. $\mathcal{H}_1 = \emptyset$. Then, $\mathcal{H}_2 = \mathcal{H}$. The computation of the partial Jacobian matrix with respect to the following variables

$$(x_h, \lambda_h, \mu_h, e_h, a_h)_{h\in\mathcal{H}}$$

is described below.

	x_h	λ_h	μ_h	e_h	a_h
$\widetilde{F}^{(h.1)}$	$D^2_{x_h} u_h(x^*_h)+\mu^*_h D^2_{x_h}\chi_h(x^*_h,e^*_h)$	$-p^{*T}$	$D_{x_h}\chi_h(x^*_h,e^*_h)^T$	$\mu^*_h D^2_{e_h x_h}\chi_h(x^*_h,e^*_h)$	
$\widetilde{F}^{(h.2)}$	$-p^*$			p^*	
$\widetilde{F}^{(h.3)}_{\mathcal{I}}$	$D_{x_h}\chi_h(x^*_h,e^*_h)$			$D_{e_h}\chi_h(x^*_h,e^*_h)$	1
$\widetilde{F}^M$	$[I_{C-1}\|0]$			$-[I_{C-1}\|0]$	

The correspondent partial system $\Delta \cdot D_{(\xi,e,a)}\widetilde{F}_{\mathcal{I}}(\xi^*, e^*, a^*) = 0$ is

$$\begin{cases} \Delta x_h \cdot \left(D^2_{x_h} u_h(x^*_h) + \mu^*_h D^2_{x_h}\chi_h(x^*_h, e^*_h)\right) - \Delta\lambda_h p^* + \\ \Delta\mu_h D_{x_h}\chi_h(x^*_h, e^*_h) + \Delta p^{\setminus} \cdot [I_{C-1}|0] = 0 \text{ for each } h \in \mathcal{H} \\ -\Delta x_h p^* = 0 \text{ for each } h \in \mathcal{H} \\ \Delta x_h D_{x_h}\chi_h(x^*_h, e^*_h) = 0 \text{ for each } h \in \mathcal{H} \\ \Delta x_h \cdot \left(\mu^*_h D^2_{e_h x_h}\chi_h(x^*_h, e^*_h)\right) + \Delta\lambda_h p^* + \\ \Delta\mu_h D_{e_h}\chi_h(x^*_h, e^*_h) - \Delta p^{\setminus} \cdot [I_{C-1}|0] = 0 \text{ for each } h \in \mathcal{H} \\ \Delta\mu_h = 0 \text{ for each } h \in \mathcal{H} \end{cases} \tag{16}$$

From $\widetilde{F}_{\mathcal{I}}(\xi^*, e^*, a^*) = 0$ in (11) and the above system, for each $h \in \mathcal{H}$ we get

$$D_{x_h} u_h(x^*_h)\Delta x_h = \lambda^*_h p^* \Delta x_h - \mu^*_h D_{x_h}\chi_h(x^*_h, e^*_h)\Delta x_h = 0 \tag{17}$$

From system (16) and Assumption 3, we get

$$\left[\Delta x_k \cdot \left(D^2_{x_k}\chi_k(x^*_k, e^*_k) + D^2_{e_k x_k}\chi_k(x^*_k, e^*_k)\right)\right] \Delta x_k \leq 0 \tag{18}$$

for some $k \in \mathcal{H}$. From system (16), we get also

$$\left(\Delta x_k \cdot D^2_{x_k} u_k(x^*_k)\right) \Delta x_k = -\mu^*_k \left[\Delta x_k \cdot \left(D^2_{x_k}\chi_k(x^*_k, e^*_k) + D^2_{e_k x_k}\chi_k(x^*_k, e^*_k)\right)\right] \Delta x_k$$

Then, (17), (18) and point 3 of Assumption 1 imply that $\Delta x_k = 0$, since $\mu^*_k > 0$. Since $p^{*C} = 1$, then we get $\Delta\lambda_k = 0$ and $\Delta p^{\setminus} = 0$. Therefore, the relevant equations of system (16) become

$$\begin{cases} \Delta x_h \cdot \left(D^2_{x_h} u_h(x^*_h) + \mu^*_h D^2_{x_h}\chi_h(x^*_h, e^*_h)\right) - \Delta\lambda_h p^* = 0 \text{ for each } h \neq k \\ -\Delta x_h p^* = 0 \text{ for each } h \neq k \\ \Delta x_h D_{x_h}\chi_h(x^*_h, e^*_h) = 0 \text{ for each } h \neq k \end{cases}$$

From the above system, we get

$$\left(\Delta x_h \cdot D^2_{x_h} u_h(x^*_h)\right) \Delta x_h = -\mu^*_h \left(\Delta x_h \cdot D^2_{x_h}\chi_h(x^*_h, e^*_h)\right) \Delta x_h \text{ for each } h \neq k$$

From (17), using similar arguments as in the proof of Lemma 18, we get $\Delta x_h = 0$ for each $h \neq k$. Finally, we get $\Delta\lambda_h = 0$ for each $h \neq k$ since $p^* \gg 0$, and then $\Delta = 0$. ■

Proof of Lemma 22. Let $u \in \mathcal{U}$. First, observe that for each $\chi \in \mathcal{X}$ there is a dense subset Λ^*_χ of $\mathbb{R}^{CH}_{++} \times \mathbb{R}^H_+$ such that $\mathcal{E}^a = (e, \chi^a, u)$ is regular for each $(e, a) \in \Lambda^*_\chi$. Indeed, as a consequence of Theorem 13 we have that for each $\chi \in \mathcal{X}$, the set Λ^r_χ of $(e, a) \in \Lambda_\chi$ such that $\mathcal{E}^a = (e, \chi^a, u)$ is regular is an open and dense subset of Λ_χ.[21] By Remark 12, $\mathbb{R}^{CH}_{++} \times \mathbb{R}^H_+ \subseteq \Lambda_\chi$ for each $\chi \in \mathcal{X}$. Then, $\Lambda^*_\chi := \Lambda^r_\chi \cap (\mathbb{R}^{CH}_{++} \times \mathbb{R}^H_+)$ is a dense subset of $\mathbb{R}^{CH}_{++} \times \mathbb{R}^H_+$.

Now, suppose otherwise that there are $(\bar{e}, \bar{\chi}) \in \mathbb{R}^{CH}_{++} \times \mathcal{X}$, an open neighborhood I of $\bar{e}$ in $\mathbb{R}^{CH}_{++}$ and an open neighborhood N of $\bar{\chi}$ in $\mathcal{X}$ such that $\mathcal{E} = (e, \chi, u)$ is not regular for every $(e, \chi) \in I \times N$. Without loss of generality we suppose that $N = \{\chi \in \mathcal{X} : \forall\, h \in \mathcal{H},\ d(\chi_h, \bar{\chi}_h) < \varepsilon_h\}$ where $0 < \varepsilon_h < 1$ for each $h \in \mathcal{H}$. It is easy to check that $\bar{\chi} + a \in N$, for each $a = (a_h)_{h\in\mathcal{H}}$ with $0 \leq a_h < \varepsilon_h$. Then, we can conclude that there is an open subset A of $\mathbb{R}^{CH}_{++} \times \mathbb{R}^H_+$ such that $\mathcal{E}^a = (e, \chi^a, u)$ is not regular for every $(e, a) \in A$. This is a contradiction since $\Lambda^*_{\bar{\chi}}$ is a dense subset of $\mathbb{R}^{CH}_{++} \times \mathbb{R}^H_+$. ■

Proof of Lemma 23. Let $u \in \mathcal{U}$. We want to prove that C_1 is sequentially closed in $F^{-1}(0)$. Let $(\xi^v, e^v, \chi^v)_{v\in\mathbb{N}} \subseteq C_1$ be a sequence converging to $(\bar{\xi}, \bar{e}, \bar{\chi}) \in F^{-1}(0)$. We have to show that $(\bar{\xi}, \bar{e}, \bar{\chi}) \in C_1$. Otherwise, suppose that $F_{\bar{\mathcal{E}}}$ is differentiable in $\bar{\xi}$ where $\bar{\mathcal{E}} := (\bar{e}, \bar{\chi}, u)$. Then, for each $h \in \mathcal{H}$, either

$$\bar{\mu}_h > 0 \text{ or } \bar{\chi}_h(\bar{x}_h, \bar{e}_h) > 0$$

Define $\mathcal{H}_1 := \{h \in \mathcal{H} : \bar{\mu}_h > 0\}$ and $\mathcal{H}_2 := \{h \in \mathcal{H} : \bar{\chi}_h(\bar{x}_h, \bar{e}_h) > 0\}$. For each $h \in \mathcal{H}_1$ there is v_h such that $\mu^v_h > 0$ for each $v \geq v_h$, and for each $h \in \mathcal{H}_2$, for given $0 < \varepsilon_h < \bar{\chi}_h(\bar{x}_h, \bar{e}_h)$, there is n_h such that $\bar{\chi}_h(x^v_h, e^v_h) > \varepsilon_h$ for each $v \geq n_h$. Since $(\chi^v)_{v\in\mathbb{N}}$ converges uniformly on compacta and the set $\{(x^v_h, e^v_h)_{v\in\mathbb{N}}\} \cup \{(\bar{x}_h, \bar{e}_h)\}$ is compact, we have that for each $h \in \mathcal{H}_2$ there is m_h such that for each $m \geq m_h$, $\chi^m_h(x^v_h, e^v_h) > \bar{\chi}_h(x^v_h, e^v_h) - \varepsilon_h$ for each $v \in \mathbb{N}$. Now, take $\bar{v} = \max\limits_{h\in\mathcal{H}}\{v_h, n_h, m_h\}$, we have that for each $v \geq \bar{v}$

$$\mu^v_h > 0, \forall\, h \in \mathcal{H}_1 \text{ and } \chi^v_h(x^v_h, e^v_h) > 0, \forall\, h \in \mathcal{H}_2$$

That is, given $\mathcal{E}^v := (e^v, \chi^v, u)$, $F_{\mathcal{E}^v}$ is differentiable in ξ^v for each $v \geq \bar{v}$, which is a contradiction. ■

Proof of Lemma 24. Let $u \in \mathcal{U}$. We have to show that any sequence $(\xi^v, e^v, \chi^v)_{v\in\mathbb{N}} \subseteq F^{-1}(0)$, up to a subsequence, converges to an element of

[21] Observe that for each $\chi \in \mathcal{X}$, Λ_χ is defined as the set Λ given in Definition 11.

$F^{-1}(0)$, knowing that $(e^v, \chi^v)_{v \in \mathbb{N}}$, up to a subsequence, converges to $(\bar{e}, \bar{\chi}) \in \mathbb{R}^{CH}_{++} \times \mathcal{X}$.

As in the proof of Lemma 17, the sequence $(x^v)_{v \in \mathbb{N}}$, up to a subsequence, converges to $\bar{x} \geq 0$. The main difficulty is to prove that for each $h \in \mathcal{H}$, $\bar{x}_h \gg 0$. Using similar arguments as in the proof of Proposition 29, it easy to show that there exists a continuous selection function $\widetilde{x}_h : \mathbb{R}^C_{++} \to \mathbb{R}^C_{++}$ such that for each $e_h \in \mathbb{R}^C_{++}$, $\bar{\chi}_h(\widetilde{x}_h(e_h), e_h) > 0$ and $\widetilde{x}_h(e_h) \ll e_h$. Then, in particular

$$\bar{\chi}_h(\widetilde{x}_h(\bar{e}_h), \bar{e}_h) > 0 \text{ and } \widetilde{x}_h(\bar{e}_h) \ll \bar{e}_h$$

and

$$\bar{\chi}_h(\widetilde{x}_h(e^v_h), e^v_h) > 0 \text{ and } \widetilde{x}_h(e^v_h) \ll e^v_h$$

for each $v \in \mathbb{N}$. Now, we are going to show that there is $\bar{v}$ such that for each $v \geq \bar{v}$

$$\chi^v_h(\widetilde{x}_h(e^v_h), e^v_h) > 0 \tag{19}$$

Since $\widetilde{x}_h$ is continuous, for given $0 < \varepsilon_h < \bar{\chi}_h(\widetilde{x}_h(\bar{e}_h), \bar{e}_h)$, there is n_h such that $\bar{\chi}_h(\widetilde{x}_h(e^v_h), e^v_h) > \varepsilon_h$ for each $v \geq n_h$. Since $(\chi^v)_{v \in \mathbb{N}}$ converges uniformly on compacta and the set $\{(\widetilde{x}_h(e^v_h), e^v_h))_{v \in \mathbb{N}}\} \cup \{(\widetilde{x}_h(\bar{e}_h), \bar{e}_h)\}$ is compact, we have that there is m_h such that for each $m \geq m_h$, $\chi^m_h(\widetilde{x}_h(e^v_h), e^v_h) > \bar{\chi}_h(\widetilde{x}_h(\bar{e}_h), \bar{e}_h) - \varepsilon_h$ for each $v \in \mathbb{N}$. Now, take $\bar{v} = \max\{n_h, m_h\}$, we have that for each $v \geq \bar{v}$, (19) holds true.

Then, by $F^{h.1}(\xi^v, e^v, \chi^v) = 0$ and $F^{h.2}(\xi^v, e^v, \chi^v) = 0$ in (5), $u_h(x^v_h) \geq u_h(\widetilde{x}_h(e^v_h))$ for every $v \geq \bar{v}$. Define $\mathbf{1} := (1, ..., 1) \in \mathbb{R}^C_{++}$, from point 2 of Assumption 1 we have that for each $\varepsilon > 0$, $u_h(x^v_h + \varepsilon\mathbf{1}) \geq u_h(\widetilde{x}_h(e^v_h))$ for every $v \geq \bar{v}$. So taking the limit on v, since u_h and $\widetilde{x}_h$ are continuous, for each $\varepsilon > 0$ we get $u_h(\bar{x}_h + \varepsilon\mathbf{1}) \geq u_h(\widetilde{x}_h(\bar{e}_h)) := \underline{u}_h$. By point 4 of Assumption 1, $\bar{x}_h \in \mathbb{R}^C_{++}$ since $\bar{x}_h$ belongs to the set $cl_{\mathbb{R}^C}\{x_h \in \mathbb{R}^C_{++} : u_h(x_h) \geq \underline{u}_h\}$.

The remaining part of the proof follows the same steps as in the proof of Lemma 17. ■

Proof of Lemma 25. Let $u \in \mathcal{U}$. For each $(\xi^*, e, \chi) \in C_2$, the determinant of all the square submatrices of $D_\xi F_\mathcal{E}(\xi^*)$ of dimension $\dim \Xi$ is equal to zero. Since the determinant is a continuous function, C_2 is closed in $F^{-1}(0)$. ■

Proof of Lemma 26. The proof follows the same steps as in the proof of Lemma 24. ■

Appendix B

The theory of general economic equilibrium from a differentiable prospective is based on results from differential topology. Following are the ones used in our analysis. These results, as well as generalizations on these issues, can be found for instance in Mas-Colell (1985) and Villanacci et al. (2002).

Topology of the C^2 uniform convergence on compacta. Denote $X := \mathbb{R}^{2C}_{++}$. Let $C^2(X,\mathbb{R})$ be the set of C^2 functions from X to $\mathbb{R}$. Endow $C^2(X,\mathbb{R})$ with the topology of *the C^2 uniform convergence on compacta.* That is, a sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges to f if and only if $(f_n)_{n\in\mathbb{N}}$, $(Df_n)_{n\in\mathbb{N}}$ and $(D^2f_n)_{n\in\mathbb{N}}$ converge uniformly to f, Df and D^2f respectively, on any compact $K \subseteq X$. Since X is contained in the closure of its interior, $C^2(X,\mathbb{R})$ is metrizable (see Mas-Colell, 1985, p. 50). Note that $C^2(X,\mathbb{R})$ can be made metric space in the following way: let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact subset of X such that $\bigcup_{n\in\mathbb{N}} K_n = X$, define the metric d by

$$d(f,g) := \sum_{n\in\mathbb{N}} \frac{1}{2^n} \min\{\|f-g\|_{2,K_n}, 1\}$$

for f and g in $C^2(X,\mathbb{R})$, where $\|\cdot\|_{1,K_n}$ is the C^2 uniform norm on $C^2(K_n,\mathbb{R})$ defined by

$$\|v\|_{2,K_n} := \sup_{x\in K_n} |v(x)| + \sup_{x\in K_n} \|Dv(x)\| + \sup_{x\in K_n} \left\|D^2v(x)\right\|$$

for $v \in C^2(K_n,\mathbb{R})$. The topology generated by d on $C^2(X,\mathbb{R})$ coincides with the topology of the C^2 uniform convergence on compacta. [22]

Theorem 30 *(Regular value theorem) Let M, N be C^r manifolds of dimensions m and n, respectively. Let $f: M \to N$ be a C^r function. Assume $r > \max\{m-n, 0\}$. If $y \in N$ is a regular value for f, then*

(1) if $m < n$, $f^{-1}(y) = \emptyset$,
(2) if $m \geq n$, either $f^{-1}(y) = \emptyset$, or $f^{-1}(y)$ is an $(m-n)$-dimensional submanifold of M.

Corollary 31 *Let M, N be C^r manifolds of the same dimension. Let $f: M \to N$ be a C^r function. Assume $r \geq 1$. Let $y \in N$ a regular value for f such that $f^{-1}(y)$ is non-empty and compact. Then, $f^{-1}(y)$ is a finite subset of M.*

The following results is a consequence of Sard's Theorem for manifolds.

Theorem 32 *Let M, Ω and N be C^r manifolds of dimensions m, p and n, respectively. Let $f: M\times\Omega \to N$ be a C^r function. Assume $r > \max\{m-n, 0\}$. If $y \in N$ is a regular value for f, then there exists a full measure subset Ω^* of*

[22] See Allen (1981) and Mas-Colell (1985).

Ω *such that for any* $\omega \in \Omega^*$, $y \in N$ *is a regular value for* f_ω, *where*

$$f_\omega : \xi \in M \to f_\omega(\xi) := f(\xi, \omega) \in N$$

Definition 33 *Let* (X, d) *and* (Y, d') *be two metric spaces. A function* $\pi : X \to Y$ *is proper if it is continuous and one among the following conditions holds true.*

(1) π *is closed and* $\pi^{-1}(y)$ *is compact for each* $y \in Y$,
(2) if K *is a compact subset of* Y, *then* $\pi^{-1}(K)$ *is a compact subset of* X,
(3) if $(x^n)_{n\in\mathbb{N}}$ *is a sequence in* X *such that* $(\pi(x^n))_{n\in\mathbb{N}}$ *converges, then* $(x^n)_{n\in\mathbb{N}}$ *has a converging subsequence in* X.

Observe that the above conditions are equivalent.

Corollary 34 *Let* M, Ω *and* N *be* C^r *manifolds of dimensions* m, p *and* n, *respectively. Let* $f : M \times \Omega \to N$ *be a* C^r *function. Assume* $r > \max\{m-n, 0\}$. *Let* Γ *be a full measure subset of* Ω *such that for any* $\omega \in \Gamma$, $y \in N$ *is a regular value for* f_ω. *If the projection* $\pi_\Omega : (\xi, \omega) \in f^{-1}(y) \to \pi_\Omega(\xi, \omega) := \omega \in \Omega$ *is proper, then* Γ *is open in* Ω.

Theorem 35 *(Implicit Function Theorem) Let* M, N *be* C^r *manifolds of the same dimension. Assume* $r \geq 1$. *Let* (X, τ) *be a topological space, and* $f : M \times X \to N$ *be a continuous function such that* $D_\xi f(\xi, x)$ *exists and it is continuous on* $M \times X$. *If* $f(\xi, x) = 0$ *and* $D_\xi f(\xi, x)$ *is onto, then there exist an open neighborhood* I *of* x *in* X, *an open neighborhood* U *of* ξ *in* M *and a continuous function* $g : I \to U$ *such that* $g(x) = \xi$ *and* $f(\xi', x') = 0$ *holds for* $(\xi', x') \in U \times I$ *if and only if* $\xi' = g(x')$.

References

Arrow, K. J., Debreu, G., 1954. Existence of an equilibrium for a competitive economy. Econometrica 22, 265-290.

Allen, B., 1981. Utility perturbations and the equilibrium price set. Journal of Mathematical Economics 8, 277-307.

Balasko, Y., 1988. Foundations of the General Equilibrium Theory. Academic Press, Boston.

Balasko, Y., Cass, D., Siconolfi, P., 1990. The structure of financial equilibrium with exogenous yields: the case of restricted participation. Journal of Mathematical Economics 19, 195-216.

Bonnisseau, J.-M., Rivera Cayupi, J., 2003. The equilibrium manifold with boundary constraints on the consumption sets. Estudios de Economía 30, 225-240.

Bonnisseau, J.-M., Rivera Cayupi, J., 2006. Constrained consumptions, Lipshitzian demands and regular economies. Journal of Optimization Theory and Applications 131, 179-193.

Cass, D., 1990. Real indeterminacy from imperfect financial markets: two addenda. CARESS Working Paper, University of Pennsylvania.

Cass, D., Siconolfi, P., Villanacci, A., 2001. Generic regularity of competitive equilibria with restricted participation. Journal of Mathematical Economics 36, 61-76.

Citanna, A., Kajii, A., Villanacci, A., 1998. Constrained suboptimality in incomplete markets: a general approach and two applications. Economic Theory 11, 495-522.

Debreu, G., 1959. Theory of value. Yale University Press.

Debreu, G., 1970. Economies with a finite set of equilibria. Econometrica 38, 387-392.

Debreu, G., 1976. Regular differentiable economies. American Economic Review 66, 387-392.

Debreu, G., 1983. Mathematical economics. Cambridge University Press.

Dierker, E., 1982. Regular economies, Chap. 17 in Handbook of Mathematical Economics, vol. II, ed. K. Arrow and M. Intriligator, 795-830. North-Holland.

Florenzano, M., 2003. General equilibrium analysis, existence and optimality properties of equilibria. Kluwer Academic Publishers.

Mas-Colell, A., 1985. The theory of general economic equilibrium. A differentiable approach. Cambridge University Press.

del Mercato, E.L., 2006a. Existence of competitive equilibria with externalities: A differential viewpoint. Journal of Mathematical Economics 42, 525-543.

del Mercato, E.L., 2006b. Exchange economies and consumption possibility: a proof of generic regularity. DiMaD Discussion Paper, University of Florence.

Polemarchakis, H., Siconolfi, P., 1997. Generic existence of competitive equilibria with restricted participation. Journal of Mathematical Economics 28, 289-311.

Smale, S., 1974a. Global analysis and economies IIA; Extension of a theorem of Debreu. Journal of Mathematical Economics 1, 1-14.

Smale, S., 1974b. Global analysis and economies IV; Finiteness and stability of equilibria with general consumption sets and production. Journal of Mathematical Economics 1, 119-127.

Villanacci, A., 1993. A proof of existence and regularity for an exchange economy with nonnegativity constraints. DiMaD Discussion Paper, University of Florence.

Villanacci, A., Carosi, L., Benevieri, P., Battinelli, A., 2002. Differential topology and general equilibrium with complete and incomplete markets. Kluwer Academic Publishers.

Villanacci, A., Zenginobuz, E.U., 2005. Existence and regularity of equilibria in a general equilibrium model with private provision of a public good. Journal of Mathematical Economics 41, 617-636.